CLASSIFICATION OF SIMPLE LIE ALGEBRAS ON A LATTICE

Kenji Iohara and Olivier Mathieu*

April 4, 2012

Abstract

Let $\Lambda = \mathbb{Z}^n$ for some $n \geq 1$. The aim of the paper is to classify all simple Λ-graded Lie algebras $\mathcal{L} = \bigoplus_{\lambda \in \Lambda} \mathcal{L}_\lambda$, such that $\dim \mathcal{L}_\lambda = 1$ for all λ. The classification involves two affine Lie algebras, namely $A_1^{(1)}$ and $A_2^{(2)}$, and a family (W_π), parametrized by a dense open set of the space of all embeddings $\pi : \Lambda \to \mathbb{C}^2$. The family (W_l) of generalized Witt algebras, indexed by all embeddings $l : \Lambda \to \mathbb{C}$, appears as a sub-family. In general, the algebras W_π are described as Lie algebras of symbols of twisted pseudo-differential operators.

0 Introduction

0.1 Statement of the Theorem proved in the paper

Let Λ be a lattice of rank n, i.e. $\Lambda \simeq \mathbb{Z}^n$. Following I. M. Gelfand's terminology, a *Lie algebra on the lattice* Λ is a Λ-graded Lie algebra $\mathcal{L} = \oplus_{\lambda \in \Lambda} \mathcal{L}_\lambda$ such that $\dim \mathcal{L}_\lambda = 1$ for all λ.

I.M. Gelfand (in his Seminar) and A.A. Kirillov [Ki] raised the question of the classification of all Lie algebras on a lattice. Of course, $\mathcal{L}$ should satisfy additional properties in order to expect an answer. The present paper investigates the case of simple graded Lie algebras on a lattice. Recall that the Lie algebra $\mathcal{L}$ is called *simple graded* if $\mathcal{L}$ has no non-trivial proper graded ideal (here it is assumed that $n > 0$, otherwise one has to assume that $\dim \mathcal{L} > 1$).

To our best of our knowledge, the first instance of this question is the V.G. Kac paper [Ka2], where he gave an explicit conjecture for the classification of all

---

simple $\mathbb{Z}$-graded Lie algebras $\mathcal{L} = \oplus_{n \in \mathbb{Z}} \mathcal{L}_n$ for which $\dim \mathcal{L}_n = 1$ for all n. He conjectured that such a Lie algebra is isomorphic to the loop algebras $A_1^{(1)}$, $A_2^{(2)}$ or to the Witt algebra W, see Section 0.2 for the definition of these algebras. His conjecture is now proved [M1].

However, for lattices of rank > 1, there was no explicit conjecture (however see Yu's Theorem [Y] cited below) .

The main result of the paper is the classification of all simple graded Lie algebras on a lattice. To clarify the statement, the notion of primitivity is defined. Let $\mathcal{L}$ be a simple Λ-graded Lie algebra, and let $m > 0$. The Lie algebra $\mathcal{L}(m) = \mathcal{L} \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_1^{\pm m}]$ is a simple $\Lambda \times \mathbb{Z}^m$-graded Lie algebra, and it is called an *imprimitive form* of $\mathcal{L}$. A simple Λ-graded Lie algebra $\mathcal{L}$ is called *primitive* if it is not an imprimitive form. It is clear that any simple $\mathbb{Z}^n$-graded Lie algebra is isomorphic to $\mathcal{L}(m)$ for some primitive $\mathbb{Z}^{n-m}$-graded Lie algebra $\mathcal{L}$.

In this paper, a family of Lie algebras W_π is introduced. This family is parametrized by an injective additive map $\pi : \Lambda \to \mathbb{C}^2$, and it contains all generalized Witt algebras. The result proved in this paper is the following one:

Main Theorem: *Let $\mathcal{L}$ be a primitive Lie algebra on Λ. Then $\mathcal{L}$ is isomorphic to $A_1^{(1)}$, $A_2^{(2)}$ or to some W_π, where $\pi : \Lambda \to \mathbb{C}^2$ is an injective and additive map satisfying condition $\mathcal{C}$.* $\qquad\square$

The next section of the introduction will be devoted to the precise definitions of the Lie algebras $A_1^{(1)}$, $A_2^{(2)}$ and W_π involved in the theorem (as well as the condition $\mathcal{C}$).

Since the proof of the theorem is quite long, the paper is divided into three chapters (see Section 0.4). Each chapter is briefly described in Sections 0.5-0.7. References for the definitions of the Lie algebras are given in Section 0.3, and for the tools used in the proof in Section 0.8.

0.2 Definition of the the Lie algebras involved in the classification

In what follows, the following convention will be used. The identity $\deg x = \lambda$ tacitly means that x is a homogeneous element, and its degree is λ.

The Lie algebra $A_1^{(1)}$: By definition, it is the Lie algebra $\mathfrak{sl}(2) \otimes \mathbb{C}[T, T^{-1}]$, where the $\mathbb{Z}$-gradation is defined by the following requirements:
$$\deg e \otimes T^n = 3n + 1, \quad \deg h \otimes T^n = 3n \text{ and } \deg f \otimes T^n = 3n - 1.$$
Here $\{e,\, f,\, h\}$ is the standard basis of $\mathfrak{sl}(2)$.

The Lie algebra $A_2^{(2)}$: For $x \in \mathfrak{sl}(3)$, set $\eta(x) = -x^t$, where t denotes the transposition. Define an involution θ of $\mathfrak{sl}(3) \otimes \mathbb{C}[T, T^{-1}]$ by
$$\theta(x \otimes T^n) = (-1)^n \, \eta(x) \otimes T^n,$$
for any $x \in \mathfrak{sl}(3)$ and $n \in \mathbb{Z}$. By definition, $A_2^{(2)}$ is the Lie algebra of fixed points of the involution θ.

The $\mathbb{Z}$-gradation of $A_2^{(2)}$ is more delicate to define. Let $(e_i,\ f_i\ h_i)_{i=1,\,2}$ be Chevalley's generators of $\mathfrak{sl}(3)$. Relative to these generators, we have $\eta(x_1) = x_2$ and $\eta(x_2) = x_1$, where the letter x stands for e, f or h.

Then the gradation is defined by the following requirements:

$\deg(f_1 + f_2) \otimes T^{2n} = 8n - 1$ $\qquad \deg(h_1 + h_2) \otimes T^{2n} = 8n,$

$\deg(e_1 + e_2) \otimes T^{2n} = 8n + 1$ $\qquad \deg[f_1, f_2] \otimes T^{2n+1} = 8n + 2,$

$\deg(f_1 - f_2) \otimes T^{2n+1} = 8n + 3$ $\qquad \deg(h_1 - h_2) \otimes T^{2n+1} = 8n + 4,$

$\deg(e_1 - e_2) \otimes T^{2n+1} = 8n + 5$ $\qquad \deg[e_1, e_2] \otimes T^{2n+1} = 8n + 6.$

The generalized Witt algebras W_l: Let $A = \mathbb{C}[z, z^{-1}]$ be the Laurent polynomial ring. In what follows, its spectrum $\operatorname{Spec} A = \mathbb{C}^*$ is called the *circle*. The *Witt algebra* is the Lie algebra $W = \operatorname{Der} A$ of vector fields on the circle. It has basis $(L_n)_{n \in \mathbb{Z}}$, where $L_n = z^{n+1} \dfrac{d}{dz}$, and the Lie bracket is given by:

$$[L_n, L_m] = (m - n)\, L_{n+m}.$$

Let $\mathcal{A}$ be the *twisted Laurent polynomial ring.* By definition $\mathcal{A}$ has basis $(z^s)_{s \in \mathbb{C}}$ and the product is given by $z^s . z^t = z^{s+t}$. The operator $\dfrac{d}{dz}$ extends to a derivation of $\partial : \mathcal{A} \to \mathcal{A}$ defined by $\partial z^s = s z^{s-1}$, for all $s \in \mathbb{C}$. The Lie algebra $\mathcal{W} = \mathcal{A}.\partial$ will be called the *twisted Witt algebra.* It has basis $(L_s)_{s \in \mathbb{C}}$, $L_s = z^{s+1} \dfrac{d}{dz}$ and the Lie bracket is given by:

$$[L_s, L_t] = (t - s)\, L_{s+t}.$$

For any injective additive map $l : \Lambda \to \mathbb{C}$, denote by W_l the subalgebra of $\mathcal{W}$ with basis (L_s), where s runs over the subgroup $l(\Lambda)$. Using the notation L_λ for $L_{l(\lambda)}$, the algebra W_l has basis $(L_\lambda)_{\lambda \in \Lambda}$ and the bracket is given by:

$$[L_\lambda, L_\mu] = l(\mu - \lambda)\, L_{\lambda + \mu}.$$

For the natural gradation of W_l, relative to which each L_λ is homogeneous of degree λ, W_l is Lie algebra on the lattice Λ. Moreover W_l is simple ([Y], Theorem 3.7).

The algebra W_π: Recall that the ordinary pseudo-differential operators on the circle are formal series $\sum_{n \in \mathbb{Z}} a_n \partial^n$, where $a_n \in \mathbb{C}[z, z^{-1}]$ and $a_n = 0$ for $n \gg 0$, and where $\partial = \dfrac{d}{dz}$. The definition of *twisted pseudo-differential operators* is similar, except that complex powers of z and of ∂ are allowed (for a rigorous definition, see Section 12). For $\lambda = (u, v) \in \mathbb{C}^2$, let E_λ be the symbol of $z^{u+1} \partial^{v+1}$.

Thus the algebra $\mathcal{P}$ of symbols of twisted pseudo-differential operators has basis E_λ, where λ runs over $\mathbb{C}^2$, and the Poisson bracket of symbols is given by the following formula:

$$\{E_\lambda, E_\mu\} = <\lambda + \rho \,|\, \mu + \rho>\ E_{\lambda + \mu},$$

where $< \,|\, >$ denotes the usual symplectic form on $\mathbb{C}^2$, and where $\rho = (1, 1)$.

Set $\Lambda = \mathbb{Z}^n$ and let $\pi : \Lambda \to \mathbb{C}^2$ be any injective additive map. By definition, W_π is the Lie subalgebra with basis $(E_\lambda)_{\lambda \in \pi(\Lambda)}$. The Lie algebra W_π is obviouly Λ-graded, for the requirement that each $E_{\pi(\lambda)}$ is homogeneous of degree λ.

Now consider the following condition:

$(\mathcal{C})$ $\qquad \pi(\Lambda) \not\subset \mathbb{C}\rho$ and $2\rho \notin \pi(\Lambda)$.

Under Condition $\mathcal{C}$, the Lie algebra W_π is simple, see Lemma 49. Moreover the generalized Witt algebras constitute a sub-family of the family (W_π): they correspond to the case where $\pi(\Lambda)$ lies inside a complex line of $\mathbb{C}^2$.

0.3 Some references for the Lie algebras:

In the context of the classification of infinite dimensional Lie algebras, the algebras $A_1^{(1)}$, $A_2^{(2)}$ (and all affine Lie algebras) first appeared in the work of V.G. Kac [Ka1]. At the same time, they were also introduced by R. Moody in other context [Mo].

In the context of the classification of infinite dimensional Lie algebras, the generalized Witt algebras appeared in the work of R. Yu [Y]. He considered simple graded Lie algebras $\mathcal{L} = \oplus_\lambda \mathcal{L}_\lambda$, where each homogeneous component $\mathcal{L}_\lambda$ has dimension one with basis L_λ.

In our terminology, the Yu Theorem can be restated as follows:

Theorem: (R.Yu) *Assume that the Lie bracket is given by:*
$$[L_\lambda, L_\mu] = (f(\mu) - f(\lambda))L_{\lambda+\mu},$$
for some function $f : \Lambda \to \mathbb{C}$. Then $\mathcal{L}$ is an imprimitive form of $A_1^{(1)}$ or an imprimitive form of a generalized Witt algebra.

0.4 General structure of the paper:

The paper is divided into three chapters. For $i = 1$ to 3, Chapter i is devoted to the proof of Theorem i. The Main Theorem is an immediate consequence of Theorems 1-3, which are stated below.

0.5 About Theorem 1

Let $\mathcal{G}$ be the class of all simple graded Lie algebras $\mathcal{L} = \oplus_\lambda \mathcal{L}_\lambda$ where each homogeneous component $\mathcal{L}_\lambda$ has dimension one. For $\mathcal{L} \in \mathcal{G}$, let L_λ be a basis of $\mathcal{L}_\lambda$, for each $\lambda \in \Lambda$.

We have $[L_0, L_\lambda] = l(\lambda)L_\lambda$ for some function $l : \Lambda \to \mathbb{C}$. The first step of the proof is the following alternative:

Theorem 1

(i) *The function $l : \Lambda \to \mathbb{C}$ is additive, or*

(ii) *there exists $a \in \mathbb{C}$ such that $l(\Lambda) = [-N, N].a$, for some positive integer N.*

Here $[-N, N]$ denotes the set of integers between $-N$ and N. In the first case, $\mathcal{L}$ is called *non-integrable* and in the second case $\mathcal{L}$ is called *integrable of type N*. Moreover the only possible value for the type is 1 or 2.

Thus Theorem 1 separates the proof into two cases, and for the integrable case there are some specificities to type 1 and to type 2. Some statements, like

the crucial Main Lemma, are common for all the cases, but they do not admit a unified proof.

In order to state this lemma, denote by Σ the set of all $\lambda \in \Lambda$ such that the Lie subalgebra $\mathbb{C}L_\lambda \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_{-\lambda}$ is isomorphic to $\mathfrak{sl}(2)$. Obviously, λ belongs to Σ iff

$$l(\lambda) \neq 0 \text{ and } [L_\lambda, L_{-\lambda}] \neq 0.$$

The Main Lemma is the following statement:

Main Lemma *An element $\lambda \in \Lambda$ belongs to Σ whenever $l(\lambda) \neq 0$.*

For the proof of the Main Lemma, see Lemma 6.4 for the case of integrable Lie algebras of type 2, Lemma 8.3 for the case of integrable Lie algebras of type 1 and Lemma 14.8 for the non-integrable case.

0.6 Statement of Theorem 2

Theorem 2 *Any primitive integrable Lie algebra in the class $\mathcal{G}$ is isomorphic to $A_1^{(1)}$ or $A_2^{(2)}$.*

The Main Lemma provides a lot of subalgebras isomorphic to $\mathfrak{sl}(2)$. Therefore the proof of Theorem 2 is based on basic notions, among them $\mathfrak{sl}(2)$-theory, Jordan algebra, Weyl group, centroid.

0.7 About the proof of Theorem 3

The main difficulty of the paper is the following statement:

Theorem 3 *Any primitive non-integrable Lie algebra in class $\mathcal{G}$ is isomorphic to W_π for some injective additive map $\pi : \Lambda \to \mathbb{C}^2$ satisfying condition $\mathcal{C}$.*

The first step is the proof that for any $\alpha \in \Sigma$, the Lie algebra
$$\mathcal{L}(\alpha) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{n\alpha}$$
is isomorphic to the Witt algebra W. Roughly speaking, it means that any subalgebra $\mathcal{L}_{-\alpha} \oplus \mathcal{L}_0 \oplus \mathcal{L}_\alpha$ isomorphic to $\mathfrak{sl}(2)$ "extends" to a Witt algebra, see Lemma 11.4.

For any $\mathbb{Z}\alpha$-coset β, set $\mathcal{M}(\beta) = \oplus_{n \in \mathbb{Z}} \mathcal{L}(\beta + n\alpha)$. Thanks to the Kaplansky-Santharoubane Theorem [KS], the possible W-module structures on $\mathcal{M}(\beta)$ are explicitly classified.

Moreover the Lie bracket in $\mathcal{L}$ provides some W-equivariant bilinear maps $B_{\beta,\gamma} : \mathcal{M}(\beta) \times \mathcal{M}(\gamma) \to \mathcal{M}(\beta + \gamma)$. It turns out that all W-equivariant bilinear maps $b : L \times M \to N$, where L, M and N are in the Kaplansky Santharoubane list, have been recently classified in [IM]. The whole list of [IM] is intricate, because it contains many special cases. However it allows a detailed analysis of the Lie bracket of $\mathcal{L}$.

The Main Lemma also holds in this setting, but its proof has little in common with the previous two cases.

0.8 Some references for the tools used in the proof

There are two types of tools used in the proof, "local analysis" and "global analysis".

A Λ-graded Lie algebras $\mathcal{L}$ contains a lot of $\mathbb{Z}$-graded subalgebras $L = \oplus_{n \in \mathbb{Z}} L_n$. Thus it contains some local Lie algebras $L_1 \oplus L_0 \oplus L_{-1}$, which determines some sections of $\mathcal{L}$. The notion of a local Lie algebra is due to V.G. Kac [Ka1]. Here, the novelty is the use of infinite dimensional local Lie algebras.

Global analysis means to investigate the decomposition of $\mathcal{L}$ under a Lie subalgebra $\mathfrak{s}$ and to analyse the Lie bracket as an $\mathfrak{s}$-equivariant bilinear map $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$. The prototype of global methods is Koecher-Kantor-Tits construction, which occurs when a subalgebra $\mathfrak{sl}(2)$ acts over a Lie algebra L in a such way that the non-zero isotypical components are trivial or adjoint: the Lie bracket of L is encoded by a Jordan algebra (up to some technical conditions). Indeed this tool, introduced by Tits [T] (see also [Kan], [Ko]), is used in chapter II, as the main ingredient to study type 1 integrable Lie algebras.

The global analysis in the non-integrable case (Section 17, 18 and 19) is quite similar, except that the subalgebra is the Witt algebra W. In this case, it is proved in Section 18 that the Lie bracket is encoded by a certain map $c : \Lambda \times \Lambda \to \mathbb{C}^*$. Indeed c satisfies a two-cocyle identity, except that it is valid only on a subset of Λ^3. Therefore, c is informally called a "quasi-two-cocycle".

0.9 Ground field

For simplicity, it has been fixed that the ground field is $\mathbb{C}$. However, the final result is valid on any field K of characteristic zero, but the proof requires the following modifications.

In Section 3, any norm of the $\mathbb{Q}$-vector space K should be used instead of the complex absolute value. The proof of Lemma 19.1 uses that $\mathbb{C}$ is uncountable, but it can be improved to include the case of a countable field. Also, the fact that $\mathbb{C}$ is algebraically closed is not essential, because $\mathcal{L}_0$ has dimension 1 and it can be check that every structure used in the proof is split.

Contents

Chapter I
The alternative integrable/non-integrable

1 Generalities, notations, conventions

This section contains the main definitions used throughout the paper. It also contains a few lemmas which are repeatedly used.

1.1

Given an integers a, set $\mathbb{Z}_{\geq a} = \{n \in \mathbb{Z} \mid n \geq a\}$. Similarly define $\mathbb{Z}_{>a}$, $\mathbb{Z}_{\leq a}$, and $\mathbb{Z}_{<a}$. Given two integers a, b with $a \leq b$, it is convenient to set $[a,b] = \mathbb{Z}_{\geq a} \cap \mathbb{Z}_{\leq b}$.

1.2

Let M be an abelian group. A *weakly M-graded vector space* is a vector space E endowed with a decomposition $E = \oplus_{m \in M} E_m$. The subspaces E_m are called the *homogeneous components* of E. Given two weakly M-graded vector space, a map $\psi : E \to F$ is called *homogeneous of degree l* if $\psi(E_m) \subset F_{m+l}$ for all $m \in M$.

When all homogeneous components are finite dimensional, E is called a *M-graded vector space*. When the grading group M is tacitely determined, E will be called a *graded vector space*. A subspace $F \subset E$ is *graded* if $F = \oplus_{m \in M} F_m$, where $F_m = F \cap E_m$. The set $\operatorname{Supp} E := \{m \in M \mid E_m \neq 0\}$ is called the *support* of M.

Set $E' = \oplus_{m \in M} (E_m)^*$. The space E', which is a subspace of the ordinary dual E^* of E, is called the *graded dual* of E. The graded dual E' admits a M-gradation defined by $E'_m = (E_{-m})^*$. As it is defined, the duality pairing is homogeneous of degree zero.

1.3

Let M be an abelian group. A *M-graded algebra* is M-graded vector space $A = \oplus_{m \in M} A_m$ with an algebra structure such that $A_l.A_m \subset A_{l+m}$ for all l, $m \in M$, where . denotes the algebra product. When the notion of A-module is defined, a *graded module* is a A-module $E = \oplus E_m$ such that $A_l.E_m \subset E_{m+l}$. The notion of *weakly graded algebras* and *weakly graded modules* are similarly defined.

1.4

An algebra A is called *simple* if $A.A \neq 0$ and A contains no non-trivial proper two-sided ideal. A M-graded algebra A is called *simple graded* if $A.A \neq 0$ and A contains no graded non-trivial proper two-sided ideals. Moreover the graded algebra A is called *graded simple* if A is simple (as a non-graded algebra).

1.5

Let M be an abelian group. Its *group algebra* is the algebra $C[M]$ with basis $(e^\lambda)_{\lambda \in M}$ and the product is defined by $e^\lambda e^\mu = e^{\lambda+\mu}$. This algebra has a natural M gradation, for which e^λ is homogeneous of degree λ. It is clear that $C[M]$ is a simple graded algebra, although this algebra is not simple.

1.6

Let Q and M be abelian groups, let $\pi : Q \to M$ be an additive map, and let A be a M-graded algebra. By definition $\pi^* A$ is the subalgebra of $A \otimes \mathbb{C}[Q]$ defined by:

$$\pi^* A := \oplus_{\lambda \in Q} A_{\pi(\lambda)} \otimes e^\lambda.$$

The algebra $\pi^* A$ is Q-graded and there is a natural algebra morphism $\psi : \pi^* A \to A$, $a \otimes e^\lambda \mapsto a$. Indeed for each $\lambda \in Q$, ψ induces an isomorphism $(\pi^* A)_\lambda \simeq A_{\pi(\lambda)}$. The following obvious lemma characterizes $\pi^* A$.

Lemma 1.1. *Let B be a Q-graded algebra and let $\phi : B \to A$ be an algebra morphism. Assume that $\phi(B_\lambda) \subset A_{\pi(\lambda)}$ and that the induced map $B_\lambda \to A_{\pi(\lambda)}$ is an isomorphism for each $\lambda \in Q$. Then the Q-graded algebra B is isomorphic to $\pi^* A$.*

1.7

From now on, Λ denotes a lattice, i.e. a free abelian group of finite rank.

1.8

Let M be an abelian group. Denote by $\mathcal{G}(M)$ the class of all simple M-graded Lie algebras $\mathcal{L} = \oplus_{\lambda \in M} \mathcal{L}_\lambda$ such that $\dim \mathcal{L}_\lambda = 1$ for all $\lambda \in M$.

Denote by $\mathcal{G}'(M)$ the class of all simple M-graded Lie algebras $\mathcal{L} = \oplus_{\lambda \in M} \mathcal{L}_\lambda$ such that

(i) $\dim \mathcal{L}_\lambda \leq 1$ for all $\lambda \in M$,

(ii) $\dim \mathcal{L}_0 = 1$, and

(iii) $\operatorname{Supp} \mathcal{L}$ generates M.

When it is tacitely assumed that $M = \Lambda$, these classes will be denoted by $\mathcal{G}$ and $\mathcal{G}'$.

1.9

Let $\mathcal{L} \in \mathcal{G}'$. For any $\lambda \in \operatorname{Supp} \mathcal{L}$, denotes by L_λ any non-zero vector of $\mathcal{L}_\lambda$. Also denote by L_λ^* the element of $\mathcal{L}'$ defined by

$$< L_\lambda^* | L_\mu > = \delta_{\lambda, \mu},$$

where $\delta_{\lambda, \mu}$ is Kronecker's symbol. Note that L_λ^* is a homogeneous element of $\mathcal{L}'$ of degree $-\lambda$.

For $\lambda \neq 0$, the exact normalization of L_λ does not matter. However, we will fix once for all the vector L_0. This allows to define the function $l : \operatorname{Supp} \mathcal{L} \to \mathbb{C}$ by the requirement

$$[L_0, L_\lambda] = l(\lambda) L_\lambda.$$

The following Lemma is obvious.

Lemma 1.2. *Let $\lambda, \mu \in \operatorname{Supp} \mathcal{L}$. If $[L_\lambda, L_\mu] \neq 0$, then*
$$l(\lambda + \mu) = l(\lambda) + l(\mu).$$

1.10

Let Π be the set of all $\alpha \in \Lambda$ such that $\pm\alpha \in \operatorname{Supp}\mathcal{L}$ and $[L_\alpha, L_{-\alpha}] \neq 0$. Let Σ be the set of all $\alpha \in \Pi$ such that $l(-\alpha) \neq 0$.

For $\alpha \in \Pi$, set $\mathfrak{s}(\alpha) = \mathbb{C}L_{-\alpha} \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_\alpha$. It is clear that $\mathfrak{s}(\alpha)$ is a Lie subalgebra. Since $l(-\alpha) = -l(\alpha)$, the Lie algebra $\mathfrak{s}(\alpha)$ is isomorphic with $\mathfrak{sl}(2)$ if $\alpha \in \Sigma$ and it is the Heisenberg algebra of dimension three otherwise.

1.11

The following lemma will be used repeatedly.

Lemma 1.3. *Let L be a Lie algebra and let A, B be two subspaces such that $L = A + B$ and $[A, B] \subset B$. Then $B + [B, B]$ is an ideal of L.*

The proof is easy, see [M1], Lemma 6.

1.12

Let X, Y be subsets of Λ. Say that X is *dominated* by Y, and denote it by $X \leq Y$, iff there exists a finite subset F of Λ with $X \subset F + Y$. Say that X is *equivalent* to Y, and denote it by $X \equiv Y$, iff $X \leq Y$ and $Y \leq X$.

In what follows, a *simple graded module* means a module without non-trivial graded submodule. The following abstract lemma will be useful:

Lemma 1.4. *Let L be a Λ-graded Lie algebra, and let M be a simple Λ-graded module. Then for any non-zero homogeneous elements m_1, m_2 of M, we have*
$$\operatorname{Supp} L.m_1 \equiv \operatorname{Supp} L.m_2.$$

Proof. For any homogeneous element $m \in M$, set $\Omega(m) = \operatorname{Supp} L.m$. More generally any $m \in M$ can be decomposed as $m = \sum m_\lambda$ where $m_\lambda \in M_\lambda$. In general set $\Omega(m) = \cup\Omega(m_\lambda)$.

Let $m \in M$, $x \in L$ be homogeneous elements. We have $L.x.m \subset L.m + x.L.m$, and therefore $\Omega(x.m) \subset \Omega(m) \cup \deg x + \Omega(m)$. It follows that $\Omega(x.m) \leq \Omega(m)$ for any (not necessarily homogeneous) elements $m \in M$ and $x \in L$. So the subspace X of all m such that $\Omega(m) \leq \Omega(m_1)$ is a graded submodule. Since M is simple as a graded L-module, we have $X = M$ and therefore $\Omega(m_2) \leq \Omega(m_1)$. Similarly we have $\Omega(m_1) \leq \Omega(m_2)$. Thus we get
$$\operatorname{Supp} L.m_1 \equiv \operatorname{Supp} L.m_2.$$
$\square$

2 Generalities on centroids of graded algebras

Let A be an algebra of any type (Jordan, Lie, associative...). Its *centroid*, denoted by $C(A)$, is the algebra of all maps $\psi : A \to A$ which commute with left and right multiplications, namely
$$\psi(ab) = \psi(a)b \text{ and } \psi(ab) = a\psi(b),$$
for any $a, b \in A$.

Lemma 2.1. *Assume that $A = A.A$. Then the algebra $C(A)$ is commutative.*

Proof. Let $\phi, \psi \in C(A)$ and $a, b \in A$. It follows from the definition that:
$$\phi \circ \psi(ab) = \phi(a)\psi(b) \text{ and } \phi \circ \psi(ab) = \psi(a)\phi(b).$$
Hence $[\phi, \psi]$ vanishes on $A.A$ and therefore $C(A)$ is commutative whenever $A = A.A$. $\square$

From now on, let Q be a countable abelian group.

Lemma 2.2. *Let A be a simple Q-graded algebra. Then:*

  (i) *Any non-zero homogeneous element $\psi \in C(A)$ is invertible.*

  (ii) *The algebra $C(A)$ is a Q-graded algebra.*

  (iii) *$M := \operatorname{Supp} C(A)$ is a subgroup of Q.*

  (iv) *The algebra $C(A)$ is isomorphic with the group algebra $\mathbb{C}[M]$.*

Proof. Point (i): Let $\psi \in C(A)$ be a non-zero and homogeneous. Its image and its kernel are graded two-sided ideals. Therefore ψ is bijective. It is clear that its inverse lies in $C(A)$.

Point (ii): Let $\psi \in C(A)$. Then ψ can be decomposed as $\psi = \sum_{\mu \in Q} \psi_\mu$, where:

  (i) the linear map $\psi_\mu : A \to A$ is homogeneous of degree μ,

  (ii) For any $x \in A$, $\psi_\mu(x) = 0$ for almost all μ.

It is clear that each homogeneous component ψ_μ is in $C(A)$. Since each non-zero component ψ_μ is injective, it follows that almost all of them are zero. Therefore each $\psi \in C(A)$ is a finite sum of its homogeneous components, and thus $C(A)$ admits a decomposition $C(A) = \oplus_\mu C(A)_\mu$.

It is obvious that $C(A)$ is a weakly graded algebra, i.e. $C(A)_\lambda.C(A)_\mu \subset C(A)_{\lambda+\mu}$ for all $\lambda, \mu \in Q$. So it remains to prove that each homogeneous component $C(A)_\mu$ is finite dimensional. Choose any homogeneous element $a \in A \setminus \{0\}$ of degree ν. Since any non-zero homogeneous element of $C(A)$ is one-to-one, the natural map $C(A) \to A, \psi \mapsto \psi(a)$ is injective. Therefore $\dim C(A)_\mu \leq \dim A_{\mu+\nu} < \infty$. Thus $C(A)$ is a Q-graded algebra.

Point (iii): Since any non-zero homogeneous element in $C(A)$ is invertible, M is a subgroup of Q.

Point (iv): By Lemma 5, $C(A)$ is commutative. Since $C(A)$ has countable dimension, any maximal ideal of $C(A)$ determines a morphism $\chi : C(A) \to \mathbb{C}$. Since each non-zero homogeneous element is invertible, the restriction of χ to each homogeneous component $C(A)_\mu$ is injective. Therefore $C(A)_\mu$ is one dimensional and there is a unique element $E_\mu \in C(A)_\mu$ with $\chi(E_\mu) = 1$. It follows that $(E_\mu)_{\mu \in M}$ is a basis of $C(A)$ and that $E_{\mu_1} E_{\mu_2} = E_{\mu_1+\mu_2}$. Hence $C(A)$ is isomorphic with the group algebra $\mathbb{C}[M]$. $\square$

Let $\mathcal{U} = \oplus_{\lambda \in Q} \mathcal{U}_\lambda$ be an associative weakly graded algebra. Set $\mathcal{A} = \mathcal{U}_0$.

Lemma 2.3. *Let E be a simple graded $\mathcal{U}$-module.*

   (i) *The $\mathcal{A}$-module E_λ is simple for any $\lambda \in \operatorname{Supp} E$.*

   (ii) *If the $\mathcal{A}$-module E_λ and E_μ are isomorphic for some λ, $\mu \in \operatorname{Supp} E$, then there is an isomorphism of $\mathcal{U}$-modules $\psi : E \to E$ which is homogeneous of degree $\lambda - \mu$.*

Proof. Point (i) Let x be any non-zero element in E_λ. Since $\mathcal{U}.x$ is a graded submodule of E, we have $\mathcal{U}.x = E$, i.e. $E_\mu = \mathcal{U}_{\mu-\lambda}.x$ for any $\mu \in Q$. So we get $\mathcal{A}.x = E_\lambda$, for any $x \in E_\lambda \setminus 0$, which proves that the $\mathcal{U}$-module E_λ is simple.

Intermediate step: Let $\mu \in Q$. Let $Irr(\mathcal{A})$ be the category of simple $\mathcal{A}$-modules and let $Irr_\mu(\mathcal{U})$ be the category of simple weakly graded $\mathcal{U}$-modules Y whose support contains μ. We claim that the categories $Irr(\mathcal{A})$ and $Irr_\mu(\mathcal{U})$ are equivalent.

By the first point, the map $E \mapsto E_\mu$ provides a functor
$$F^\mu : Irr_\mu(\mathcal{U}) \to Irr(\mathcal{A}).$$
Conversely, let $X \in Irr(\mathcal{A})$. Set $I^\mu(X) = \mathcal{U} \otimes_\mathcal{A} X$, and for any $\lambda \in Q$, set $I_\lambda^\mu(X) = \mathcal{U}_{\lambda-\mu} \otimes_\mathcal{A} X$. Relative to the decomposition
$$I^\mu(X) = \oplus_{\lambda \in Q} I_\lambda^\mu(M),$$
$I^\mu(X)$ is a weakly graded $\mathcal{U}$-module. Let $K(X)$ be the biggest $\mathcal{U}$-submodule lying in $\oplus_{\lambda \neq \mu} I_\lambda^\mu(M)$ and set $S^\mu(X) = I^\mu(X)/K(X)$. Since $K(X)$ is a graded subspace, $S^\mu(X)$ is a weakly graded $\mathcal{U}$-module, and it is clear that $S^\mu(X)$ is simple as weakly graded module. Moreover its homogeneous component of degre μ is X. Therefore the map $X \mapsto S^\mu(X)$ provides a functor
$$S^\mu : Irr(\mathcal{A}) \to Irr_\mu(\mathcal{U}).$$
The functor F^μ and S^μ are inverse to each other, which proves that the category $Irr(\mathcal{A})$ and $Irr_\mu(\mathcal{U})$ are equivalent.

Point (ii): Let λ, $\mu \in \operatorname{Supp} E$. Assume that the $\mathcal{U}$-modules E_λ and E_μ are isomorphic to the same $\mathcal{A}$-module X. As $\mathcal{U}$-modules, $S^\mu(X)$ and $S^\lambda(X)$ are isomorphic, up to a shift by $\lambda - \mu$ of their gradation. Since we have $E \simeq S^\mu(X)$ and $E \simeq S^\lambda(X)$, there is isomorphism of $\mathcal{U}$-modules $\psi : A \to A$ which is homogeneous of degree $\lambda - \mu$. $\qquad\blacksquare$

Let A be a graded algebra. Let $\mathcal{U} \subset \operatorname{End}(A)$ be the associative subalgebra generated by left and right multiplications. Let $\mathcal{A} \subset \mathcal{U}$ be the subalgebra of all $u \in \mathcal{U}$ such that $u(A_\mu) \subset A_\mu$ for all $\mu \in Q$.

Lemma 2.4. *Let A be a simple Q-graded algebra. Set $M = \operatorname{Supp} C(A)$.*

   (i) *For any $\mu \in \operatorname{Supp} A$, the $\mathcal{A}$-module A_μ is simple.*

   (ii) *For any λ, $\mu \in \operatorname{Supp} A$, the $\mathcal{A}$-modules A_μ and A_λ are isomorphic iff $\lambda - \mu \in M$.*

Proof. For each $\mu \in Q$, set $\mathcal{U}_\mu = \{u \in \mathcal{U} \,|\, u(A_\nu) \subset A_{\mu+\nu}, \forall \nu\}$. Relative to the decomposition $\mathcal{U} = \oplus_{\mu \in Q} \mathcal{U}_\mu$, the algebra $\mathcal{U}$ is a weakly graded, and A is a simple graded $\mathcal{U}$-module.

The simplicity of each $\mathcal{A}$-module A_μ follows from Lemma 2.3 (i).

Assume that $\lambda - \mu \in M$. Choose any non-zero $\psi \in C(A)$ which is homogeneous of degree $\lambda - \mu$. Then ψ provides a morphism of $\mathcal{A}$-modules $A_\mu \to A_\lambda$. By Lemma 2.2 (i), it is an isomorphism, so A_λ and A_μ are isomorphic.

Conversely, assume that the $\mathcal{A}$-modules A_λ and A_μ are isomorphic. By Lemma 2.3 (ii), there is a isomorphism of $\mathcal{U}$-modules $\psi : E \to E$ which is homogeneous of degree $\lambda - \mu$. Since ψ belongs to $C(A)$, it follows that $\lambda - \mu$ belongs to M. □

Lemma 2.5. *Let A be a simple Q-graded algebra, let M be the support of $C(A)$. Set $\overline{Q} = Q/M$ and let $\pi : Q \to \overline{Q}$ be the natural projection. Then there exists a $\overline{Q}$-graded algebra $\overline{A}$ such that:*

(i) $A = \pi^*\overline{A}$,

(ii) the algebra $\overline{A}$ is simple (as a non-graded algebra).

Proof. Let m be any maximal ideal of $C(A)$. Set $\overline{A} = A/m.A$ and let $\psi : A \to \overline{A}$ be the natural projection.

For $\lambda \in Q$, set $\mathcal{M}(\lambda) = \oplus_{\mu \in M} A_{\lambda+\mu}$. It follows from Lemma 2.2 that $\mathcal{M}(\lambda) = C(A) \otimes A_\lambda$, $C(A)/m \cong \mathbb{C}$ and so we have $\mathcal{M}(\lambda)/m.\mathcal{M}(\lambda) \simeq A_\lambda$. Thus $\overline{A}$ is a $\overline{Q}$-graded algebra. Moreover for each $\lambda \in Q$, the morphism ψ induces an isomorphism $A_\lambda \to \overline{A}_{\pi(\lambda)}$. It follows from Lemma 1.1 that $A \simeq \pi^*\overline{A}$.

Since ψ is onto, it is clear that $\overline{A}$ is simple graded. By Lemma 2.4, the family $(\overline{A}_{\overline{\lambda}})_{\overline{\lambda} \in \operatorname{Supp} \overline{A}}$ consists of simple, pairwise non-isomorphic $\mathcal{A}$-modules. Thus any two-sided ideal in $\overline{A}$ is graded, which proves that the algebra $\overline{A}$ is simple (as a non-graded algebra). □

3 The alternative for the class $\mathcal{G}$

Let Λ be a lattice. This section contains the first key result for the classification, namely the alternative for the class $\mathcal{G}$.

However in chapter II, the class $\mathcal{G}'$ will be used as a tool, and some results are more natural in this larger class. Except stated otherwise, $\mathcal{L}$ will be a Lie algebra in the class $\mathcal{G}'$.

Lemma 3.1. *Let $\alpha \in \Pi$ and $\lambda \in \operatorname{Supp} \mathcal{L}$. Assume $l(\lambda) \neq 0$, and set $x = |2l(\lambda)/l(\alpha)|$ if $l(\alpha) \neq 0$ or $x = +\infty$ otherwise. Then at least one of the following two assertions holds:*

(i) $\operatorname{ad}^n(L_\alpha)(L_\lambda) \neq 0$ for all $n \leq x$, or

(ii) $\operatorname{ad}^n(L_{-\alpha})(L_\lambda) \neq 0$ for all $n \leq x$.

Proof. Set $\mathfrak{s}(\alpha) := \mathbb{C}L_\alpha \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_{-\alpha}$. Recall that $\mathfrak{s}(\alpha)$ is a Lie subalgebra. Moreover $\mathfrak{s}(\alpha)$ is isomorphic to $\mathfrak{sl}(2)$ if $\alpha \in \Sigma$ or to the 3-dimensional Heisenberg algebra $\mathfrak{H}$ if $\alpha \in \Pi \backslash \Sigma$. So the lemma follows from standard representation theory of $\mathfrak{sl}(2)$ and $\mathfrak{H}$.

Here is a short proof. One can assume that $\operatorname{ad}^n(L_{\pm\alpha})(L_\lambda) = 0$ for n big enough, otherwise the assertion is obvious. Set
$$N^\pm = \operatorname{Sup}\{n | \operatorname{ad}^n(L_{\pm\alpha})(L_\lambda) \neq 0\}.$$
Let M be the $\mathfrak{s}(\alpha)$-module generated by L_λ. Then the family $(L_{\lambda+n\alpha})_{n\in[-N^-,N^+]}$ is a basis for M, and therefore M is finite dimensional. Since L_0 is a scalar mutiple of $[L_\alpha, L_{-\alpha}]$, we have $\operatorname{tr} L_0|_M = 0$. Since $[L_0, L_{\lambda+n\alpha}] = (l(\lambda) + nl(\alpha)) L_{\lambda+n\alpha}$ for all $n \in [-N^-, N^+]$, we get

$$0 = \sum_{-N^- \leq n \leq N^+} (l(\lambda) + nl(\alpha))$$
$$= (N^+ + N^- + 1)[(N^+ - N^-)l(\alpha)/2 + l(\lambda)].$$

Therefore we have $|N^+ - N^-| = 2|l(\lambda)/l(\alpha)| = x$, which proves that N^+ or N^- is $\geq x$. $\qquad\blacksquare$

Lemma 3.2. *Let $\alpha_1, \alpha_2, ..., \alpha_n \in \Pi$.*

There exists a positive real number $a = a(\alpha_1, \ldots, \alpha_n)$ such that for every $\lambda \in \operatorname{Supp}\mathcal{L}$, and every n-uple $(m_1, m_2, ..., m_n)$ of integers with $0 \leq m_i \leq a|l(\lambda)|$ for all i, there exists $(\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n$ such that:
$$\operatorname{ad}^{m_n}(L_{\epsilon_n\alpha_n}) \ldots \operatorname{ad}^{m_1}(L_{\epsilon_1\alpha_1})(L_\lambda) \neq 0.$$

Proof. First define the real number a. If $l(\alpha_i) = 0$ for all i, choose any $a > 0$. Otherwise, set $s = \operatorname{Max}|l(\alpha_i)|$ and set $a = 1/ns$. Let $\lambda \in \Lambda$. Then the following assertion:

$(\mathcal{H}_k)$: for every k-uple $(m_1, m_2, ..., m_k)$ of integers with $0 \leq m_i \leq a|l(\lambda)|$ for all i, there exists $(\epsilon_1, \ldots, \epsilon_k) \in \{\pm 1\}^k$ such that
$$\operatorname{ad}^{m_k}(L_{\epsilon_k\alpha_k}) \ldots \operatorname{ad}^{m_1}(L_{\epsilon_1\alpha_1})(L_\lambda) \neq 0.$$
will be proved by induction on k, for $0 \leq k \leq n$. Clearly, one can assume that $l(\lambda) \neq 0$.

Assume that $\mathcal{H}_k$ holds for some $k < n$. Let $(m_1, m_2, ..., m_{k+1})$ be a $(k+1)$-uple of integers with $0 \leq m_i \leq a|l(\lambda)|$ for all i. By hypothesis, there exists $(\epsilon_1, \ldots, \epsilon_k) \in \{\pm 1\}^k$ such that $X := \operatorname{ad}^{m_k}(L_{\epsilon_k\alpha_k}) \ldots \operatorname{ad}^{m_1}(L_{\epsilon_1\alpha_1})(L_\lambda)$ is not zero. Set $\mu = \lambda + \sum_{i\leq k} \epsilon_i m_i \alpha_i$. Since $\deg X = \mu$, we have $l(\mu) = l(\lambda) + \sum_{i\leq k} \epsilon_i m_i l(\alpha_i)$. It follows that

$$|l(\mu)| \geq |l(\lambda)| - \sum_{i\leq k} m_i|l(\alpha_i)|$$
$$\geq |l(\lambda)| - \sum_{i\leq k} a|l(\lambda)|$$
$$\geq |l(\lambda)| - \sum_{i\leq k} |l(\lambda)|/n$$
$$\geq (n-k)|l(\lambda)|/n$$
$$\geq |l(\lambda)|/n,$$

and therefore we have:

$$|2l(\mu)/l(\alpha_{i+1})| \geq |l(\mu)/l(\alpha_{i+1})| \geq |l(\lambda)|/ns = a|l(\lambda)|.$$

By Lemma 3.1, there exists some $\epsilon \in \{\pm 1\}$ such that $\mathrm{ad}^{\,m}(L_{\epsilon\alpha_{k+1}})(X) \neq 0$ for any $m \leq a|l(\mu)|$. Set $\epsilon_{k+1} = \epsilon$. It follows that

$$\mathrm{ad}^{\,m_{k+1}}(L_{\epsilon_{k+1}\alpha_{k+1}}) \ldots \mathrm{ad}^{\,m_1}(L_{\epsilon_1\alpha_1})(L_\lambda) \neq 0$$

Therefore Assertion $(\mathcal{H}_{k+1})$ is proved.

Since $(\mathcal{H}_0)$ is trivial, the assertion $(\mathcal{H}_n)$ is proved and the lemma follows. $\square$

Lemma 3.3. *The set Π generates Λ, and Σ is not empty.*

Proof. Let M be the sublattice generated by Π and let N be its complement. Set $\mathcal{A} = \oplus_{\lambda \in M}\mathcal{L}_\lambda$, $\mathcal{B} = \oplus_{\lambda \in N}\mathcal{L}_\lambda$ and $\mathcal{I} = \mathcal{B} + [\mathcal{B}, \mathcal{B}]$. Since $\Lambda = M \cup N$ and $M + N \subset N$, we have $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $[\mathcal{A}, \mathcal{B}] \subset \mathcal{B}$. Therefore, it follows from Lemma 1.3 that $\mathcal{I}$ is an ideal.

By hypothesis, N does not contain 0 nor any element in Π and thus $\mathcal{I} \cap \mathcal{L}_0 = \{0\}$. Since $\mathcal{I} \neq \mathcal{L}$, it follows that $\mathcal{I} = 0$. Therefore the support of $\mathcal{L}$ lies in M. Since $\mathrm{Supp}\,\mathcal{L}$ generates Λ, it is proved that Π generates Λ.

The proof of the second assertion is similar. Set $\mathcal{A} = \oplus_{l(\lambda)=0}\mathcal{L}_\lambda$, $\mathcal{B} = \oplus_{l(\lambda)\neq 0}\mathcal{L}_\lambda$ and $\mathcal{I} = \mathcal{B} + [\mathcal{B}, \mathcal{B}]$. We have $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and by Lemma 1.2 we have $[\mathcal{A}, \mathcal{B}] \subset \mathcal{B}$. Therefore $\mathcal{I}$ is an ideal. Since L_0 is not central, this ideal is not trivial and so $\mathcal{I} = \mathcal{L}$. It follows easily that L_0 belongs to $[\mathcal{B}, \mathcal{B}]$, thus there is $\alpha \in \Lambda$ such that $l(\alpha) \neq 0$ and $[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \neq 0$. By definition α belongs to Σ, so Σ is not empty. $\square$

A subset $B \in \Lambda$ is called a $\mathbb{Q}$-*basis* if it is a basis of $\mathbb{Q} \otimes \Lambda$. Equivalently, it means that B is a basis of a finite index sublattice in Λ. From the previous lemma, Π contains some $\mathbb{Q}$-basis B. Then define the additive map $L_B : \Lambda \to \mathbb{C}$ by the requirement that $L_B(\beta) = l(\beta)$ for all $\beta \in B$.

Lemma 3.4. *Assume that l is an unbounded function. Then we have $L_B(\alpha) = l(\alpha)$, for any $\alpha \in \Pi$.*

Proof. For clarity, the proof is divided into four steps.

Step one: Additional notations are now introduced.

Let $\| \, \|$ be a euclidean norm on Λ, i.e. the restriction of a usual norm on $\mathbb{R} \otimes \Lambda$. For any positive real number r, the ball of radius r is the set $B(r) = \{\lambda \in \Lambda | \, \|\lambda\| \leq r\}$. There is a positive real numbers v such that Card $B(r) \leq vr^n$ for all $r \geq 1$.

Fix $\alpha \in \Pi$. Set $B = \{\alpha_1, \ldots, \alpha_n\}$, where n is the rank of Λ and set $\alpha_{n+1} = \alpha$. Let $a = a(\alpha_1, \ldots, \alpha_{n+1})$ be the constant of Lemma 3.2 and set $b = \sum_{1 \leq i \leq n+1} \|\alpha_i\|$. Also for $s \geq 0$, let $A(s)$ be the set of all $(n+1)$-uples $\mathbf{m} = (m_1, \ldots, m_{n+1})$ of integers with $0 \leq m_i \leq s$ for any i.

Step two: There exists $r_0 > 1/ab$ such that $(ar)^{n+1} > v(abr)^n$ for any $r > r_0$. Equivalently, we have:

$$\mathrm{Card}\ A(ar) > \mathrm{Card}\ B(abr)$$

for any $r > r_0$.

Step three: Let $\lambda \in \operatorname{Supp}\mathcal{L}$. Define two maps: $\epsilon_\lambda : A(a|l(\lambda)|) \to \{\pm 1\}^{n+1}$ and $\Theta_\lambda : A(a|l(\lambda)|) \to B(ab|l(\lambda)|)$, by the following requirement. By Lemma 3.2, for each $(n+1)$-uple $\mathbf{m} = (m_1, \ldots, m_{n+1}) \in A(a|l(\lambda)|)$, there exists $(\epsilon_1, \ldots, \epsilon_{n+1}) \in \{\pm 1\}^{n+1}$ such that

$$\operatorname{ad}^{m_{n+1}}(L_{\epsilon_{n+1}\alpha_{n+1}}) \ldots \operatorname{ad}^{m_1}(L_{\epsilon_1\alpha_1})(L_\lambda) \neq 0.$$

Thus set

$\epsilon_\lambda(\mathbf{m}) = (\epsilon_1, ,\ldots, \epsilon_{n+1})$, and $\Theta_\lambda(\mathbf{m}) = \sum_{1\leq i\leq n+1} \epsilon_i m_i \alpha_i$.
From the definition of the maps ϵ_λ and Θ_λ, it follows that
$(*)$ $\qquad \operatorname{ad}^{m_{n+1}}(L_{\epsilon_{n+1}\alpha_{n+1}}) \ldots \operatorname{ad}^{m_1}(L_{\epsilon_1\alpha_1})(L_\lambda) = cL_{\lambda+\Theta_\lambda(\mathbf{m})},$
where c is a non-zero scalar.
Moreover, we have

$$\begin{aligned}
||\Theta_\lambda(\mathbf{m})|| &\leq \sum_{1\leq i\leq n+1} m_i||\alpha_i|| \\
&\leq a|l(\lambda)| \sum_{1\leq i\leq n+1} ||\alpha_i|| \\
&\leq ab|l(\lambda)|,
\end{aligned}$$

and therefore Θ_λ takes value in $B(ab|l(\lambda)|)$.

Step four: Since the function l is unbounded, one can choose $\lambda \in \operatorname{Supp}\mathcal{L}$ such that $|l(\lambda)| > r_0$.

It follows from Step two that Θ_λ is not injective. Choose two distinct elements $\mathbf{m}, \mathbf{m}' \in A(a|l(\lambda)|)$ with $\Theta_\lambda(\mathbf{m}) = \Theta_\lambda(\mathbf{m}')$. Set $\mathbf{m} = (m_1, \ldots, m_{n+1})$, $\mathbf{m}' = (m'_1, \ldots, m'_{n+1})$, $\epsilon_\lambda(\mathbf{m}) = (\epsilon_1, \ldots, \epsilon_{n+1})$ and $\epsilon_\lambda(\mathbf{m}') = (\epsilon'_1, \ldots, \epsilon'_{n+1})$. Using Identity $(*)$, we have:

$$\begin{aligned}
l(\lambda + \Theta_\lambda(\mathbf{m})) &= l(\lambda) + \sum_{1\leq k\leq n+1} \epsilon_i m_i l(\alpha_i) \\
&= l(\lambda) + L_B(\mu) + \epsilon_{n+1} m_{n+1} l(\alpha_{n+1}),
\end{aligned}$$

where $\mu = \sum_{1\leq k\leq n} \epsilon_i m_i \alpha_i$. Similarly, we get
$$l(\lambda + \Theta_\lambda(\mathbf{m}')) = l(\lambda) + L_B(\mu') + \epsilon'_{n+1} m'_{n+1} l(\alpha_{n+1}),$$
where $\mu' = \sum_{1\leq k\leq n} \epsilon'_i m'_i \alpha_i$. Therefore, we get:

$$(\epsilon_{n+1} m_{n+1} - \epsilon'_{n+1} m'_{n+1}) l(\alpha_{n+1}) + L_B(\mu - \mu') = 0.$$
$$(\epsilon_{n+1} m_{n+1} - \epsilon'_{n+1} m'_{n+1}) \alpha_{n+1} + \mu - \mu' = 0.$$

Since $\{\alpha_1, \ldots, \alpha_n\}$ is a $\mathbb{Q}$-basis, we have $\epsilon_{n+1} m_{n+1} \neq \epsilon'_{n+1} m'_{n+1}$. It follows from the previous two identities that $l(\alpha_{n+1}) = L_B(\alpha_{n+1})$, which proves the lemma. $\square$

A function $m : \operatorname{Supp}\mathcal{L} \to \mathbb{C}$ is called *additive* if there is an additive function $\widetilde{m} : \Lambda \to \mathbb{C}$ whose restriction to $\operatorname{Supp}\mathcal{L}$ is m. Since $\operatorname{Supp}\mathcal{L}$ generates Λ, the function M is uniquely determined by m.

Lemma 3.5. *Assume that l is an unbounded function. Then the function l is additive.*

16

Proof. It follows from the previous lemma that there exists an additive function $L : \Lambda \to \mathbb{C}$ such that $l(\alpha) = L(\alpha)$ for any $\alpha \in \Pi$.

Set $M = \{\lambda \in \operatorname{Supp} \mathcal{L} \,|\, l(\lambda) = L(\lambda)\}$ and let N be its complement. Set $\mathcal{A} = \oplus_{\lambda \in M} \mathcal{L}_\lambda$, $\mathcal{B} = \oplus_{\lambda \in N} \mathcal{L}_\lambda$ and $\mathcal{I} = \mathcal{B} + [\mathcal{B}, \mathcal{B}]$. We have $\mathcal{L} = \mathcal{A} + \mathcal{B}$. Since $l(\lambda + \mu) = l(\lambda) + l(\mu)$ whenever $[L_\lambda, L_\mu] \neq 0$ (Lemma 1.2), we also have $[\mathcal{A}, \mathcal{B}] \subset \mathcal{B}$. Therefore, it follows from Lemma 1.3 that $\mathcal{I}$ is an ideal.

By assumption, N does not contains 0, nor any element in Π and thus $\mathcal{I} \cap \mathcal{L}_0 = \{0\}$. Since $\mathcal{I} \neq \mathcal{L}$, it follows that $\mathcal{I} = 0$. Therefore $N = \emptyset$, which implies that l is additive. $\qquad\square$

An algebra $\mathcal{L} \in \mathcal{G}'$ is called *integrable* if the function $l : \operatorname{Supp} \mathcal{L} \to \mathbb{C}$ is bounded (it is similar to the definition of integrability in [M1],[M2]). Otherwise the Lie algebra $\mathcal{L}$ will be called *non-integrable*.

Recall that Σ is the set of all α such that the Lie algebra $\mathfrak{s}(\alpha) = \mathcal{L}_{-\alpha} \oplus \mathcal{L}_0 \oplus \mathcal{L}_\alpha$ is isomorphic to $\mathfrak{sl}(2)$.

Lemma 3.6. *Assume that $\mathcal{L} \in \mathcal{G}'$ is integrable. For any $\beta \in \Sigma$, we have:*
$$l(\Lambda) \subset \mathbb{Z}.l(\beta)/2.$$

Proof. Since $\beta \in \Sigma$, there is a $\mathfrak{sl}(2)$-triple (e, f, h) with $e \in \mathcal{L}_\beta$, $f \in \mathcal{L}_{-\beta}$ and $h = 2L_0/l(\beta)$. Since $\mathcal{L}$ is integrable, it is a direct sum of $\mathfrak{s}(\beta)$-modules of finite dimension and the eigenvalues of h are integers. Thus the eigenvalues of L_0 are integral multiple of $l(\beta)/2$. $\qquad\square$

Lemma 3.7. *Assume that $\mathcal{L} \in \mathcal{G}'$ is integrable. There exists $\alpha \in \Sigma$, and an integer $N \in \mathbb{Z}_{>0}$ such that:*
$$l(\Lambda) = [-N, N].l(\alpha).$$

Proof. First we claim that $\operatorname{Im} l \subset \mathbb{Z}.l(\alpha)$ for some $\alpha \in \Sigma$.

By Lemma 3.3, $\Sigma \neq \emptyset$. Choose any $\beta \in \Sigma$. If $\operatorname{Im} l \subset \mathbb{Z}.l(\beta)$, the claim is proved.

Otherwise, set $A = \{\lambda \in \operatorname{Supp} \mathcal{L} \,|\, l(\lambda) \in \mathbb{Z}.l(\alpha)\}$ and let B be its complement. Also set $\mathcal{A} = \oplus_{\lambda \in A} \mathcal{L}_\lambda$ and $\mathcal{B} = \oplus_{\lambda \in B} \mathcal{L}_\lambda$. It is clear that $\mathcal{L} = \mathcal{A} \oplus \mathcal{B}$ and $[\mathcal{A}, \mathcal{B}] \subset \mathcal{B}$. By Lemma 1.3, $\mathcal{B} + [\mathcal{B}, \mathcal{B}]$ is an ideal, therefore there exists $\alpha \in B$ such that $[L_\alpha, L_{-\alpha}]$ is a non-zero multiple of L_0. Since $l(\alpha) \notin \mathbb{Z}.l(\beta)$, we have $l(\alpha) \neq 0$, and so α lies in Σ.

By Lemma 3.6, there are integers m and n such that $l(\beta) = ml(\alpha)/2$ and $l(\alpha) = nl(\beta)/2$, which implies $mn = 4$. Moreover, $n/2$ is not an integer. Thus $n = \pm 1$ and $l(\alpha) = \pm l(\beta)/2$. By Lemma 3.6, $\operatorname{Im} l \subset \mathbb{Z}.l(\beta)/2 = \mathbb{Z}.l(\alpha)$, and therefore the claim is proved.

Since l is bounded, there a finite set $X \subset \mathbb{Z}$ such that $\operatorname{Im} l = X.l(\alpha)$. Since $\mathfrak{s}(\alpha)$ is isomorphic to $\mathfrak{sl}(2)$, $\mathcal{L}$ is a direct sum of finite dimensional simple $\mathfrak{s}(\alpha)$-modules. Thus it follows that X is necessarily a symmetric interval $[-N, N]$. $\quad\square$

Let $\mathcal{L} \in \mathcal{G}'$ be integrable. The *type* of $\mathcal{L}$ is the integer N such that $l(\Lambda) = [-N, N].l(\alpha)$ for some $\alpha \in \Sigma$.

Theorem 1. *(Alternative for the class $\mathcal{G}$) Let $\mathcal{L}$ be a Lie algebra in the class $\mathcal{G}$. Then $\mathcal{L}$ satisfies one of the following two assertions:*

(i) *The function $l : \Lambda \to \mathbb{C}$ is additive, or*

(ii) *there exists $\alpha \in \Sigma$ such that $l(\Lambda) = [-N, N].l(\alpha)$, for some positive integer N.*

Proof. Theorem 1 follows from Lemmas 3.5 and 3.7. □

Chapter II

Classification of integrable Lie algebras in $\mathcal{G}$

4 Notations and conventions for chapter II

Let Λ be a lattice. Let $\mathcal{L}$ be an integrable Lie algebra in the class $\mathcal{G}'$ of type N. By definition, the spectrum of $\mathrm{ad}\,(L_0)$ is $[-N, N].x$, for some scalar $x \in \mathbb{C}^*$. After a renormalization of L_0, it can be assumed that $x = 1$.

Set $\mathcal{L}^i = \oplus_{l(\beta)=i} \mathcal{L}_\beta$. There is a decomposition:
$$\mathcal{L} = \oplus_{i \in [-N, +N]} \mathcal{L}^i.$$
Relative to this decomposition, $\mathcal{L}$ is weakly $\mathbb{Z}$-graded: in general, the homogeneous components $\mathcal{L}^i$ are infinite dimensional. For any integer i, set $\Sigma_i = \{\beta \in \Sigma|\, l(\beta) = i\}$. Similarly, there is a decomposition
$$\Sigma = \cup_{i \in [-N, +N]} \Sigma_i.$$

5 The Lie subalgebra $\mathcal{K}$

Let e, f, h be the standard basis of $\mathfrak{sl}(2)$. For a finite dimensional $\mathfrak{sl}(2)$-module V, set $V^i = \{v \in V | h.v = 2iv\}$. A simple finite dimensional $\mathfrak{sl}(2)$-module is called *spherical* if $V^0 \neq 0$, or, equivalently if the eigenvalues of h are even integers, or, equivalently if $\dim V$ is odd. For a spherical module V, the elements of V^0 are called the *spherical vectors* of V.

Let $\pi : \mathfrak{sl}(2) \to \mathbb{C}$ defined by $\pi(h) = 1$, $\pi(e) = \pi(f) = 0$. Let U, V be two spherical simple finite dimensional $\mathfrak{sl}(2)$-modules, and let $b : U \otimes V \to \mathfrak{sl}(2)$ be a non-zero $\mathfrak{sl}(2)$-morphism. Define $B : U \times V \to \mathbb{C}$ by $B(u, v) = \pi(b(u \otimes v))$. Denote respectively by $K(U) \subset U$ and $K(V) \subset V$ the left and the right kernel of the bilinear map B.

Lemma 5.1. *(with the previous notations). Assume that $b \neq 0$ and $\dim V \geq \dim U$. Then one of the following statements holds:*

(i) *We have $\dim U = \dim V$, $K(U) = U^0$ and $K(V) = V^0$, or*

(ii) *we have $\dim U = 2m - 1$, $\dim V = 2m + 1$, $K(U) = 0$ and $K(V) = V^m \oplus V^{-m}$ for some positive integer m.*

18

Proof. It follows easily from the tensor product decomposition formula for $\mathfrak{sl}(2)$-modules that $U \otimes V$ contains the adjoint module iff $\dim U = \dim V$ or if $\dim U = \dim V - 2$. These dimensions are odd because U and V are spherical, so we have

case 1: $\dim U = \dim V$, or

case 2: $\dim U = 2m - 1$, $\dim V = 2m + 1$, for some positive integer m.

Moreover, the adjoint module appears with multiplicity one, so b is uniquely defined up to a scalar multiple.

In the first case, identify V with U^* and $\mathfrak{sl}(2)$ with its dual. Up to a non-zero scalar multiple, b can be identified with the map $U \otimes V \to \mathfrak{sl}(2)^*$ defined as follows:
$$b(u \otimes v) \text{ is the linear map } x \in \mathfrak{sl}(2) \mapsto < x.u|v >,$$
for any $u \in U$, $v \in V$, where $< \ | \ >$ is the duality pairing between U and V, and where $x.u$ denotes the action of x on u. So, up to a non-zero scalar multiple, we have $B(u, v) = < h.u|v >$. It follows easily that the left kernel of B is the kernel of $h|_U$, namely U^0. By symmetry, the right kernel of B is V^0.

In the second case, let $E = \mathbb{C}X \oplus \mathbb{C}Y$ be the two dimensional representation of $\mathfrak{sl}(2)$, where X and Y denotes two eigenvectors for h. Identify U with $S^{2(m-1)}E$, V with $S^{2m}E$ and $\mathfrak{sl}(2)$ with the dual of S^2E, where $S^l E$ denotes the space of degree l homogeneous polynomials in X and Y. Up to a non-zero scalar multiple, b can be identified with the map $S^{2(m-1)}E \otimes S^{2(m+1)}E \to (S^2E)^*$ defined as follows:
$$b(F \otimes G) \text{ is the linear map } H \in S^2E \mapsto < H.F|G >,$$
for any $F \in S^{2(m-1)}E$, $G \in S^{2(m+1)}E$, where $< \ | \ >$ is the $\mathfrak{sl}(2)$-invariant pairing on $S^{2(m+1)}E$, and where $H.F$ denotes product of polynomials H and F. So, up to a non-zero scalar multiple, we have $B(F, G) = < XYF|G >$. Since the multiplication by XY is injective, it follows that $K(U) = 0$. Since B is h-invariant, it follows that the right kernel is generated by X^{2m} and Y^{2m}, i.e. $K(V) = V^m \oplus V^{-m}$. $\qquad\blacksquare$

Set $\Lambda_e = \Lambda \times \mathbb{Z}$. Then $\mathcal{L}$ admits the natural Λ_e-gradation:
$$\mathcal{L} = \bigoplus_{(\beta,i) \in \Lambda_e} \mathcal{L}_\beta^i,$$

where $\mathcal{L}_\beta^i = \mathcal{L}_\beta \cap \mathcal{L}^i$. Set $\operatorname{Supp}_e \mathcal{L} = \{(\beta, i) \in \Lambda_e | \mathcal{L}_\beta^i \neq 0\}$.

For $\alpha \in \Sigma_1$, define $\pi_\alpha : \Lambda_e \to \Lambda$ by $\pi_\alpha(\beta, i) = \beta - i\alpha$. For $\gamma \in \Lambda$, set $\mathcal{M}(\alpha, \gamma) = \oplus_{\pi_\alpha(\beta,i)=\gamma} \mathcal{L}_\beta^i$. It is clear that $[\mathcal{M}(\alpha, \gamma_1), \mathcal{M}(\alpha, \gamma_2)] \subset \mathcal{M}(\alpha, \gamma_1 + \gamma_2)$, for any γ_1, $\gamma_2 \in \Lambda$. Since $\operatorname{Supp}_e \mathcal{L} \subset \Lambda \times [-N, N]$, it follows that $\dim \mathcal{M}(\alpha, \gamma) \leq 2N + 1$ for all γ.

Therefore, the decomposition
$$\mathcal{L} = \bigoplus_{\gamma \in \Lambda} \mathcal{M}(\alpha, \gamma)$$

provides a new Λ-gradation of $\mathcal{L}$. It is clear that each $\mathcal{M}(\alpha, \gamma)$ is an $\mathfrak{s}(\alpha)$-module.

19

Lemma 5.2. *Let $\alpha \in \Sigma_1$.*

(i) *$\mathcal{M}(\alpha, \gamma)$ is not zero iff γ belongs to $\operatorname{Supp} \mathcal{L}^0$.*

(ii) *For any $\gamma \in \operatorname{Supp} \mathcal{L}^0$, the $\mathfrak{s}(\alpha)$-module $\mathcal{M}(\alpha, \gamma)$ is simple and it is generated by the spherical vector L_γ.*

(iii) *In particular, we have $\mathcal{M}(\alpha, 0) = \mathfrak{s}(\alpha)$.*

Proof. Since α belongs to Σ_1, any simple component of the $\mathfrak{s}(\alpha)$-module $\mathcal{L}$ is spherical. Assume that $\mathcal{M}(\alpha, \gamma)$ is not zero. Then its spherical part is $\mathcal{L}_\gamma$, and so γ belongs to $\operatorname{Supp} \mathcal{L}^0$. Moreover its spherical part has dimension one, so $\mathcal{M}(\alpha, \gamma)$ is simple. Thus Points (i) and (ii) are proved.

Since $\mathcal{M}(\alpha, 0)$ contains $\mathfrak{s}(\alpha)$, it follows that $\mathcal{M}(\alpha, 0) = \mathfrak{s}(\alpha)$. $\qquad\square$

Let $B : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ be the skew-symmetric bilinear form defined by $B(X, Y) = L_0^*([X, Y])$, for any $X, Y \in \mathcal{L}$. Its kernel, denoted $\mathcal{K}$, is a Lie subalgebra. Set $\mathcal{K}^i = \mathcal{K} \cap \mathcal{L}^i$ for any integer i.

Lemma 5.3. *Let $\alpha \in \Sigma_1$ and $\gamma \in \Lambda$. Assume that:*
$$[\mathcal{M}(\alpha, \gamma), \mathcal{M}(\alpha, -\gamma)] \neq 0.$$
Then one of the following two assertions holds:

(i) *$\dim \mathcal{M}(\alpha, \gamma) = \dim \mathcal{M}(\alpha, -\gamma)$, or*

(ii) *$\dim \mathcal{M}(\alpha, \epsilon\gamma) = 3$ and $\dim \mathcal{M}(\alpha, -\epsilon\gamma) = 1$, for some $\epsilon = \pm 1$.*

Proof. Assume that neither Assertion (i) nor (ii) holds. Moreover, it can be assumed, without loss of generality, that $\dim \mathcal{M}(\alpha, \gamma) > \dim \mathcal{M}(\alpha, -\gamma)$.

By Lemma 5.2, the modules $\mathcal{M}(\alpha, \pm\gamma)$ are spherical and $\mathcal{M}(\alpha, -\gamma)$ is not the trivial representation. Thus the hypotheses imply that
$$\dim \mathcal{M}(\alpha, -\gamma) \geq 3 \text{ and } \dim \mathcal{M}(\alpha, \gamma) \geq 5.$$
Set $\beta = \gamma + \alpha$ and $\delta = \gamma + 2\alpha$. Since L_γ is a spherical vector of $\mathcal{M}(\alpha, \gamma)$, it follows that γ, β and δ belongs to $\operatorname{Supp} \mathcal{M}(\alpha, \gamma)$. Similarly, $-\gamma$ and $-\beta$ belong to $\operatorname{Supp} \mathcal{M}(\alpha, -\gamma)$.

It follows from Lemma 5.1 that $L_{-\beta}$ is not in the kernel of B. Thus β belongs to Σ_1. Since $\gamma + \delta = 2\beta$ and $l(\gamma) + l(\delta) = 2$, the element $[L_\gamma, L_\delta]$ is homogeneous of degree $(2\beta, 2)$ relative to the Λ_e-gradation. Thus $[L_\gamma, L_\delta]$ belongs to $\mathcal{M}(\beta, 0)$. By Lemma 5.2, we have $\mathcal{M}(\beta, 0) = \mathfrak{s}(\beta)$, and therefore
$$[L_\gamma, L_\delta] = 0.$$
Similarly, $[L_{-\gamma}, L_\delta]$ is homogeneous of degree $(2\alpha, 2)$ relative to the Λ_e-gradation. Thus it belongs to $\mathcal{M}(\alpha, 0)$. Since $\mathcal{M}(\alpha, 0) = \mathfrak{s}(\alpha)$, it follows that
$$[L_{-\gamma}, L_\delta] = 0.$$
However, these two relations $[L_{\pm\gamma}, L_\delta] = 0$ are impossible. Indeed it follows from Lemma 5.1 that $L_{-\gamma}$ is not in the kernel of B. Thus $[L_{-\gamma}, L_\gamma] = cL_0$ for some $c \neq 0$, and thus $[[L_{-\gamma}, L_\gamma], L_\delta] = 2cL_\delta \neq 0$, which is a contradiction. $\quad\square$

Lemma 5.4. *Let $\alpha \in \Sigma_1$. Then we have:*

(i) *$[\mathcal{L}_{-\alpha}, \mathcal{K}^i] \subset \mathcal{K}^{i-1}$ for any $i > 1$,*

(ii) $[\mathcal{L}_\alpha, \mathcal{K}^i] \subset \mathcal{K}^{i+1}$ *for any* $i \geq 1$.

Proof. For any $\gamma \in \Lambda$ and any $i \in \mathbb{Z}$, set $\mathcal{K}(\gamma) = \mathcal{M}(\alpha,\gamma) \cap \mathcal{K}$ and $\mathcal{K}^i(\gamma) = \mathcal{K}^i \cap \mathcal{K}(\gamma)$. Since $\mathcal{K}$ is a graded subspace of $\mathcal{L}$, we have $\mathcal{K} = \oplus_\gamma \mathcal{K}(\gamma)$. Therefore, it is enough to prove for any $\gamma \in \Lambda$ the following assertion:

($\mathcal{A}$) $[\mathcal{L}_{-\alpha}, \mathcal{K}^{i+1}(\gamma)] \subset \mathcal{K}^i(\gamma)$ and $[\mathcal{L}_\alpha, \mathcal{K}^i(\gamma)] \subset \mathcal{K}^{i+1}(\gamma)$, for any $i \geq 1$

Four cases are required to check the assertion.

First case: Assume $[\mathcal{M}(\alpha,\gamma), \mathcal{M}(\alpha,-\gamma)] = 0$.

In such a case, $\mathcal{K}(\gamma) = \mathcal{M}(\alpha,\gamma)$ is a $\mathfrak{s}(\alpha)$-module, and Assertion ($\mathcal{A}$) is obvious.

From now on, it can be assumed that
$$[\mathcal{M}(\alpha,\gamma), \mathcal{M}(\alpha,-\gamma)] \neq 0$$

Second case: Assume $\dim \mathcal{M}(\alpha,\gamma) = \dim \mathcal{M}(\alpha,-\gamma)$.

In such a case, it follows from Lemma 5.1 that $\mathcal{K}(\gamma) = \mathcal{K}^0(\gamma)$, and Assertion ($\mathcal{A}$) is clear.

Third case: Assume $\dim \mathcal{M}(\alpha,\gamma) < \dim \mathcal{M}(\alpha,-\gamma)$.

It follows from Lemma 5.1 that $\mathcal{K}(\gamma) = 0$ and Assertion ($\mathcal{A}$) is obvious.

Fourth case: Assume $\dim \mathcal{M}(\alpha,\gamma) > \dim \mathcal{M}(\alpha,-\gamma)$.

It follows from Lemma 5.3 that $\dim \mathcal{M}(\alpha,\gamma) = 3$ and that $\dim \mathcal{M}(\alpha,-\gamma) = 1$. Thus $\mathcal{K}(\gamma) = \mathcal{K}^1(\gamma) \oplus \mathcal{K}^{-1}(\gamma)$. Thus Assertion ($\mathcal{A}$) follows from the fact that 1 is the highest eigenvalue of L_0 on $\mathcal{M}(\alpha,\gamma)$ and therefore $[\mathcal{L}_\alpha, \mathcal{K}^1(\gamma)] = 0$.   □

Lemma 5.5. *We have:*

(i) $[\mathcal{L}^{-1}, \mathcal{K}^i] \subset \mathcal{K}^{i-1}$ *for any* $i > 1$,

(ii) $[\mathcal{L}^1, \mathcal{K}^i] \subset \mathcal{K}^{i+1}$ *for any* $i \geq 1$.

Proof. We have $\mathcal{L}^{-1} = \mathcal{K}^{-1} \oplus [\oplus_{\alpha \in \Sigma_1} \mathcal{L}_{-\alpha}]$ and $\mathcal{L}^1 = \mathcal{K}^1 \oplus [\oplus_{\alpha \in \Sigma_1} \mathcal{L}_\alpha]$. Therefore the lemma follows from the previous lemma and the fact that $\mathcal{K}$ is a Lie subalgebra.   □

6   Types of integrable Lie algebras in the class $\mathcal{G}'$

In this section, it is proved that an integrable Lie algebra $\mathcal{L} \in \mathcal{G}'$ is of type 1 or 2. In the terminology of root graded Lie algebras, it corresponds with A_1 and BC_1 Lie algebras, see in particular [BZ].

Lemma 6.1. *We have* $\Sigma_1 \neq \emptyset$ *and* $\Sigma_i = \emptyset$ *for any* $i > 2$.

Proof. The fact that $\Sigma_1 \neq \emptyset$ follows from Lemma 3.7.

Let $\beta \in \Sigma_i$ with $i > 0$. Fix $\alpha \in \Sigma_1$. By Lemma 3.6, we have $1 = l(\alpha) \in \mathbb{Z}.l(\beta)/2$. It follows that i divides 2, and therefore $i = 1$ or 2.   □

In what follows, it will be convenient to set $\mathcal{L}^{>0} = \oplus_{i>0} \mathcal{L}^i$. Similarly define $\mathcal{L}^{\geq 0}$ and $\mathcal{L}^{<0}$.

Lemma 6.2. *Let* $\mathcal{L} \in \mathcal{G}'$ *be integrable of type* N.

(i) *The Lie algebra $\mathcal{L}^{>0}$ (respectively $\mathcal{L}^{<0}$) is generated by $\mathcal{L}^1$ (respectively by $\mathcal{L}^{-1}$).*

(ii) *The commutant of $\mathcal{L}^1$ is $\mathcal{L}^N$.*

(iii) *The $\mathcal{L}^0$-module $\mathcal{L}^N$ is simple Λ-graded.*

(iv) $\mathcal{L}^k = [\mathcal{L}^{-1}, \mathcal{L}^{k+1}]$, *and* $\mathcal{L}^{k+1} = [\mathcal{L}^1, \mathcal{L}^k]$, *for any* $k \in [-N, N-1]$.

Proof. First prove Assertion (i). Let $\alpha \in \Sigma_1$. Since $\mathcal{L}$ is a direct sum of finite dimensional spherical $\mathfrak{s}(\alpha)$-modules, we have $\mathcal{L}^{k+1} = \mathrm{ad}^k(\mathcal{L}_\alpha)(\mathcal{L}^1)$ for any $k > 0$. Since $\mathcal{L}_\alpha \subset \mathcal{L}^1$, the Lie algebra $\mathcal{L}^{>0}$ is generated by $\mathcal{L}^1$. Similarly, the Lie algebra $\mathcal{L}^{<0}$ is generated by $\mathcal{L}^{-1}$. Assertion (i) is proved.

Now prove Assertion (ii). Let $\mathcal{M} = \oplus \mathcal{M}^i$ be the commutant of $\mathcal{L}^1$, where $\mathcal{M}^i = \mathcal{L}^i \cap \mathcal{M}$.

Let $i < N$ be any integer, and set $\mathcal{I} = \mathrm{Ad}\,(U(\mathcal{L}))(\mathcal{M}^i)$. By Point (i), $\mathcal{L}^{\geq 0}$ is generated by $\mathcal{L}^1$ and $\mathcal{L}^0$, thus $\mathcal{M}^i$ is a $\mathcal{L}^{\geq 0}$-submodule. Hence by PBW theorem we get $\mathcal{I} = \mathrm{Ad}\,(U(\mathcal{L}^{<0}))(\mathcal{M}^i)$. So any homogeneous component of $\mathcal{I}$ has degree $\leq i$ and $\mathcal{I}$ intersects trivially $\mathcal{L}^N$. Therefore $\mathcal{I}$ is a proper ideal. Since $\mathcal{I}$ is clearly Λ-graded, it follows that $\mathcal{I} = 0$ which implies that $\mathcal{M}^i = 0$. Therefore $\mathcal{M}$ is homogeneous of degree N. Since $\mathcal{L}^N$ is obviously in the commutant of $\mathcal{L}^1$, Assertion (ii) is proved.

The proof of Assertion (iii) is similar. Choose any Λ-graded $\mathcal{L}^0$-submodule $\mathcal{N}$ in $\mathcal{L}^N$ and set $\mathcal{I} = \mathrm{Ad}\,(U(\mathcal{L}))(\mathcal{N})$. By the same argument, it follows that $\mathcal{I}^N = \mathcal{N}$. Since the graded ideal $\mathcal{I}$ is not zero, it follows that $\mathcal{N} = \mathcal{L}^N$, i.e. the $\mathcal{L}^0$-module $\mathcal{L}^N$ is simple Λ-graded, which proves Assertion (iii).

It follows from the previous considerations that $\mathcal{L} = \mathrm{Ad}\,(U(\mathcal{L}^{<0}))(\mathcal{L}^N)$. Since $\mathcal{L}^{<0}$ is generated by $\mathcal{L}^{-1}$, we have
$$\mathcal{L} = \oplus_{k \geq 0} \mathrm{ad}^k(\mathcal{L}^{-1})(\mathcal{L}^N).$$
Similarly, we have
$$\mathcal{L} = \oplus_{k \geq 0} \mathrm{ad}^k(\mathcal{L}^1)(\mathcal{L}^{-N}).$$
Assertion (iv) follows from these two identities. $\qquad\square$

Lemma 6.3. *One of the following assertions holds:*

(i) $\mathcal{L}$ *is of type* 1*, or*

(ii) $\mathcal{L}$ *is of type* 2*, and* $\Sigma_2 \neq \emptyset$.

Proof. Let N be the type of $\mathcal{L}$. Also set $M = 2$ if $\Sigma_2 \neq \emptyset$ and set $M = 1$ otherwise. Thus Lemma 6.3 is equivalent to $N = M$.

Assume otherwise, i.e. $M < N$. It follows from the hypothesis that:
$$\mathcal{L}^{M+1} = \mathcal{K}^{M+1} \text{ and } \mathcal{L}^M \neq \mathcal{K}^M.$$
However by Lemma 6.2, we get $\mathcal{L}^M = [\mathcal{L}^{-1}, \mathcal{L}^{M+1}]$, and so $\mathcal{L}^M = [\mathcal{L}^{-1}, \mathcal{K}^{M+1}]$. From Lemma 5.5, we have $[\mathcal{L}^{-1}, \mathcal{K}^{M+1}] \subset \mathcal{K}^M$. So we obtain $\mathcal{L}^M \subset \mathcal{K}^M$ which contradicts $\mathcal{L}^M \neq \mathcal{K}^M$. $\qquad\square$

Lemma 6.4. *(Main Lemma for type* 2 *integrable Lie algebras)*
Let $\mathcal{L} \in \mathcal{G}'$ be integrable of type 2*. Then any $\beta \in \mathrm{Supp}\,\mathcal{L}$ with $l(\beta) \neq 0$ belongs to Σ.*

Proof. First step: By Lemma 6.2, we get $\mathcal{L}^0 = [\mathcal{L}^1, \mathcal{L}^{-1}]$. Since $\mathcal{L}^3 = 0$, we deduce that $[\mathcal{L}^0, \mathcal{K}^2] = [\mathcal{L}^1, [\mathcal{L}^{-1}, \mathcal{K}^2]]$. It follows from Lemma 5.5 that
$$[\mathcal{L}^1, [\mathcal{L}^{-1}, \mathcal{K}^2]] \subset [\mathcal{L}^1, \mathcal{K}^1] \subset \mathcal{K}^2$$
Hence $\mathcal{K}^2$ is an $\mathcal{L}^0$-submodule of $\mathcal{L}^2$. By the previous lemma, we have $\Sigma_2 \neq \emptyset$, so $\mathcal{K}^2$ is a proper submodule of $\mathcal{L}^2$. It follows from Lemma 6.2 (iii) that $\mathcal{K}^2 = 0$. Equivalently, any $\beta \in \operatorname{Supp}\mathcal{L}$ with $l(\beta) = 2$ belongs to Σ.

Second step: By Lemma 5.5, we have $\operatorname{ad}(\mathcal{L}^1)(\mathcal{K}^1) \subset \mathcal{K}^2$. Since $\mathcal{K}^2 = 0$, it follows that $\mathcal{K}^1$ lies inside the commutant of $\mathcal{L}^1$. However by Lemma 6.2 (ii), this commutant is $\mathcal{L}^2$. Hence we have $\mathcal{K}^1 = 0$, or, equivalently, any $\beta \in \operatorname{Supp}\mathcal{L}$ with $l(\beta) = 1$ belongs to Σ.

Final step: Similarly, it is proved that any $\beta \in \operatorname{Supp}\mathcal{L}$ with $l(\beta) = -1$ or $l(\beta) = -2$ belongs to Σ. Since l takes value in $[-2, 2]$, the Lemma is proved. $\square$

The Main Lemma also holds for type 1 integrable Lie algebras. Unfortunately, the proof requires more computations. The simplest approach, based on Jordan algebras, is developed in the next two sections. Another approach (not described in the paper) is based on tensor products of Chari-Pressley loop modules [C], [CP] for $A_1^{(1)}$.

7 Jordan algebras of the class $\mathcal{J}$

Let J be a Jordan algebra. In what follows, it is assumed that any Jordan algebra is unitary. For any $X \in J$, denote by $M_X \in \operatorname{End}(J)$ the multiplication by X and set $[X, Y] = [M_X, M_Y]$. It follows from the Jordan identity that $[X, Y]$ is a derivation of J, see [J]. Any linear combinations of such expressions is called an *inner derivation*, and the space of inner derivations of J is denoted by $\operatorname{Inn} J$.

Lemma 7.1. *Let $X, Y \in J$.*

  (i) $[X^2, Y] = 2[X, XY]$,

  (ii) $X^2 Y^2 + 2(XY)(XY) = Y(X^2 Y) + 2X(Y(YX))$.

Proof. To obtain both identities [J], apply $d\dfrac{d}{dt}\Big|_{t=0}$ to the Jordan identities:

$$[(X + tY)^2, X + tY] = 0,$$
$$(X + tY)^2(Y(X + tY)) = ((X + tY)^2 Y)(X + tY).$$

$\square$

Let Q be a finitely generated abelian group, let $J = \oplus_\lambda J_\lambda$ be a Q-graded Jordan algebra, and set $\mathcal{D} = \operatorname{Inn} J$. There is a decomposition $\mathcal{D} = \oplus_\lambda \mathcal{D}_\lambda$, where $\mathcal{D}_\lambda$ is the space of linear combinations $[X, Y]$, where $X \in J_\mu$, $Y \in J_\nu$ for some $\mu, \nu \in Q$ with $\mu + \nu = \lambda$. Relative to this decomposition, $\mathcal{D}_\lambda$ is a weakly Q-graded Lie algebra.

The Jordan algebra J is called of *class* $\mathcal{J}(Q)$ if the following requirements hold:

(i) J is a simple Q-graded Jordan algebra,

(ii) $\dim J_\lambda + \dim \mathcal{D}_\lambda \leq 1 \; \forall \, \lambda \in Q$, and

(iii) $\operatorname{Supp} J$ generates Q.

If $\pi : Q \to \overline{Q}$ is a surjective morphism of abelian groups and if $\overline{J}$ is a $\overline{Q}$-graded Jordan algebra, then $\overline{J}$ is in the class $\mathcal{J}(\overline{Q})$ if and only if $\pi^*\overline{J}$ is in $\mathcal{J}(Q)$. This follows from the functoriality of inner derivations. For now on, fix an abelian group Q and set $\mathcal{J} = \mathcal{J}(Q)$.

An element X of a Jordan algebra J is called *strongly invertible* if $XY = 1$ and $[X, Y] = 0$, for some $Y \in J$. Obviously, a strongly invertible element is invertible in the sense of [J], Definition 5.

An element $Z \in J$ is called *central* iff $[Z, X] = 0$ for any $X \in J$. The subalgebra of central elements is called the center of J. Since J is unitary, the map $\psi \mapsto \psi(1)$ identifies the centroid of J with its center. Therefore both algebras will be denoted by $C(J)$.

In a Jordan algebra of the class $\mathcal{J}$, we have $ab = 0$ or $[a, b] = 0$ for any two homogeneous elements a, b. For example, if we have $XY = 1$ for two homogeneous elements X and Y, then X is strongly invertible.

Lemma 7.2. *Let $J \in \mathcal{J}$ be a Jordan algebra, and let $X \in J$ be a homogeneous and strongly invertible element. Then X^2 is central.*

Proof. In order to show that $[X^2, Z] = 0$ for all Z, it can be assumed that Z is a non-zero homogeneous element.

First assume that $XZ = 0$. In such a case, we have
$$[X^2, Z] = 2[X, XZ] = 0,$$
by Point (i) of the previous lemma.

Assume otherwise. By hypothesis, there exists a homogeneous element $Y \in J$ such that $XY = 1$. Then we get:
$$Y(XZ) = [Y, X].Z + X(YZ) = [Y, X].Z + [X, Z].Y + Z(XY).$$
We have $[X, Y] = 0$ and since $XZ \neq 0$, we also have $[X, Z] = 0$. Therefore $Y(XZ) = Z(XY) = Z$. Since M_X and M_Y commutes, we have $Y(Y(X(XZ))) = Y(X(Y(XZ))) = Y(XZ) = Z$ and therefore $X(XZ) \neq 0$. Since $[X^2, Z]$ and $X(XZ)$ are homogeneous of the same degree, we have $[X^2, Z] = 0$ by Condition (ii) of the definition of the class $\mathcal{J}$. $\square$

Lemma 7.3. *Let J be a Jordan algebra in the class $\mathcal{J}$. Then*
$$\mathcal{D}.J \cap J_0 = \{0\}.$$

Proof. Assume otherwise. Then there are three homogeneous elements $X, Y, Z \in J$ such that $[X, Y].Z = 1$.

It can be assumed that J is generated by X, Y and Z. Indeed let J' be the Jordan algebra generated by X, Y and Z. Since J' is unitary, it contains a

unique maximal graded ideal M. Then the same relation $[X, Y].Z = 1$ holds in J/M and J/M is again in the class $\mathcal{J}$.

Also, it can be assumed that $C(J) = \mathbb{C}$. Indeed, by Lemma 2.5, there exists an abelian group $\overline{Q}$, a surjective morphism $\pi : Q \to \overline{Q}$ and a $\overline{Q}$ graded simple Jordan algebra $\overline{J}$ such that $J = \pi^* \overline{J}$. It has been noted that $\overline{J}$ is in $\mathcal{J}(\overline{Q})$ and by simplicity its centroid $C(\overline{J})$ is reduced to $\mathbb{C}$. Since the same relation $[X, Y].Z = 1$ holds in $\overline{J}$, it can be assumed that $C(J) = \mathbb{C}$.

Let α, β, γ be the degree of X, Y and Z. Note that $\alpha + \beta + \gamma = 0$.

We have $X(YZ) - Y(XZ) = 1$. By symmetry, one can assume that $X(YZ) \neq 0$.

We have $X(YZ) = [X, Z].Y + Z(XY)$. Since $[X, Y]$ is not zero, we have $XY = 0$ and $0 \neq X(YZ) = [X, Z].Y$. Thus we have $[X, Z] \neq 0$ and $XZ = 0$.

Thus $X(YZ) = 1$. It follows from Lemma 7.2 and the hypothesis $C(J) = \mathbb{C}$ that X^2 is a non-zero scalar c, and therefore $2\alpha = 0$ and $YZ = X/c$.

By Lemma 7.1 (ii), there is the identity
$$Y^2 Z^2 + 2(YZ)^2 = Z(Y^2 Z) + 2Y(Z(ZY)).$$
We have
$$\deg Y^2 Z = 2\beta + \gamma = \beta - \alpha = \beta + \alpha = \deg[X, Y], \text{ and}$$
$$\deg Z(ZY) = \beta + 2\gamma = \gamma - \alpha = \gamma + \alpha = \deg[X, Z].$$

By hypothesis we have $[X, Y] \neq 0$ and it has been proved that $[X, Z] \neq 0$. Therefore $Y^2 Z = 0$ and $Z(ZY) = 0$ and the right side of the identity is zero. Thus we get $Y^2 Z^2 + 2(YZ)^2 = 0$. Since $YZ = X/c$, it follows that $Y^2 Z^2 = -2/c$. Hence Y^2 is strongly invertible. By Lemma 7.2, Y^4 is a non-zero scalar, and therefore $4\beta = 0$.

Thus the subgroup generated by α and β has order ≤ 8, and it contains $\gamma = -(\alpha + \beta)$. Therefore Supp J is finite, i.e. $\dim J < \infty$.

It follows from Jordan identity that $M_{[X, Y].Z} = [[M_X, M_Y], M_Z]$, see [J], Formula (54). Set $P = [M_X, M_Y]$ and $Q = M_Z$. Thus P and Q belongs to the associative algebra $\mathrm{End}(J)$, and they satisfy $[P, Q] = 1$, where $[,]$ denotes the ordinary Lie bracket. Since J is finite dimensional, this identity is impossible. $\qquad\square$

Lemma 7.4. *Let J be a simple graded Jordan algebra of the class $\mathcal{J}$. Then any non-zero homogeneous element is strongly invertible.*

Proof. Let $\lambda : J \to \mathbb{C}$ be defined by $\lambda(1) = 1$ and $\lambda(X) = 0$ if X is homogeneous of degree $\neq 0$.

It follows from the previous lemma that $\lambda([X, Y].Z) = 0$, or, equivalently $\lambda((XY)Z) = \lambda(X(YZ))$ for all $X, Y, Z \in J$. Thus the kernel of the bilinear map $B : J \times J \to \mathbb{C}, (X, Y) \mapsto \lambda(XY)$ is a graded ideal. Hence B is non-degenerated, which implies that any non-zero homogeneous element is strongly invertible. $\qquad\square$

Recall that the underlying grading group is denoted by Q.

Lemma 7.5. *Let J be a Jordan algebra of the class $\mathcal{J}$. Then*
$$\mathrm{Supp}\, C(J) \supset 2Q.$$

Proof. Let $\alpha \in \operatorname{Supp} J$, and let $X \in J_\alpha \setminus \{0\}$. By Lemma 7.4, X is strongly invertible and by Lemma 7.2, X^2 is central. Therefore 2α belongs to $\operatorname{Supp} C(J)$. Since $\operatorname{Supp} J$ generates Q and $\operatorname{Supp} C(J)$ is a subgroup, it follows that $\operatorname{Supp} C(J)$ contains $2Q$. $\qquad\square$

8 Kantor-Koecher-Tits construction

Let us recall the Kantor-Koecher-Tits construction, which first appeared in [T], and then in [Kan], [Ko]. It contains two parts.

To any Jordan algebra J is associated a Lie algebra, denoted by $\mathfrak{sl}(2, J)$, whose definition is as follows. As a vector space, $\mathfrak{sl}(2, J) = \mathfrak{sl}(2) \otimes J \oplus \operatorname{Inn} J$, and the bracket is defined by the following requirements:

(i) its restriction to $\operatorname{Inn} J$ is the Lie algebra structure of $\operatorname{Inn} J$,

(ii) we have: $[\delta, x \otimes a] = x \otimes \delta.a$,

(iii) we have: $[x \otimes a, y \otimes b] = [x, y] \otimes ab + k(x, y)[a, b]$, for any $\delta \in \operatorname{Inn} J$, $x, y \in \mathfrak{sl}(2)$, $a, b \in J$, where $k(x, y) = 1/2 \operatorname{tr}(xy)$ and where $[a, b]$ denotes the inner derivation $[M_a, M_b]$.

Conversely, to certain Lie algebras are associated with Jordan algebras. Indeed assume that the Lie algebra $\mathcal{L}$ contains a subalgebra $\mathfrak{sl}(2)$ with its standard basis e, f, h and moreover that
$$\mathcal{L} = \mathcal{L}^{-1} \oplus \mathcal{L}^0 \oplus \mathcal{L}^1,$$
where $\mathcal{L}^i = \{x \in \mathcal{L} \mid [h, x] = 2ix\}$. Define an algebra J by the following requirements:

(i) as a vector space, $J = \mathcal{L}^1$,

(ii) the product of two elements x, y is given by the formula $xy = 1/2[[f, x], y]$.

Lemma 8.1 (Tits [T]). *With the previous hypothesis:*

(i) J is a Jordan algebra.

(ii) If moreover $\mathcal{L}^0 = [\mathcal{L}^1, \mathcal{L}^{-1}]$ and if the center of $\mathcal{L}$ is trivial, then we have $\mathcal{L} = \mathfrak{sl}(2, J)$.

An *admissible datum* is a triple (J, Λ', α) with the following conditions:

(i) Λ' is a subgroup of Λ, α is an element of Λ, and the group Λ is generated by Λ' and α,

(ii) J is a Jordan algebra in the class $\mathcal{J}(\Lambda')$,

(iii) the four subsets $\operatorname{Supp} J$, $\pm\alpha + \operatorname{Supp} J$ and $\operatorname{Supp} \operatorname{Inn} J$ are disjoint.

Then a Λ-gradation of the Lie algebra $\mathfrak{sl}(2, J)$ is defined as follows:

(i) On $\operatorname{Inn} J$, the gradation is the natural Λ' gradation

(ii) For any homogeneous element $x \in J$, set
$$\deg h \otimes x = \deg x, \ \deg e \otimes x = \alpha + \deg x \text{ and } \deg f \otimes x = -\alpha + \deg x.$$

The condition (iii) of an admissible triple ensures that the associated Λ-gradation of $\mathfrak{sl}(2, J)$ is mutiplicity free. Since the Λ-graded Lie algebra $\mathfrak{sl}(2, J)$ is clearly simple graded, it is a type 1 integrable Lie algebra of the class $\mathcal{G}'$.

Conversely, let Λ be a finitely generated abelian group, and let $\mathcal{L} \in \mathcal{G}'(\Lambda)$ be a type 1 integrable Lie algebra. Let Λ' be the subgroup generated by $\beta - \gamma$, where β and γ run over $\operatorname{Supp} \mathcal{L}^1$.

By Lemma 6.1, the set Σ_1 is not empty. Choose $\alpha \in \Sigma_1$. Let $J(\alpha)$ be the Jordan algebra defined by the following requirements:

(i) As a vector space, $J(\alpha) = \mathcal{L}^1$,

(ii) the gradation of $J(\alpha)$ is given by
$$J(\alpha)_\mu = \mathcal{L}^1_{\mu + \alpha},$$

(iii) the product xy of two elements $x, y \in J(\alpha)$ is defined by
$$xy = [[L_{-\alpha}, x], y].$$

Lemma 8.2. *Let $\mathcal{L} \in \mathcal{G}'$ be a type 1 integrable Lie algebra, and let $\alpha \in \Sigma_1$.*

(i) The triple $(J(\alpha), \Lambda', \alpha)$ is an admissible datum, and

(ii) as a Λ-graded Lie algebra, $\mathcal{L}$ is isomorphic to $\mathfrak{sl}(2, J(\alpha))$.

Proof. First check that $\mathcal{L}$ satisfies the hypothesis of the previous lemma. By simplicity of $\mathcal{L}$, its center is trivial. Moreover the identity $\mathcal{L}^0 = [\mathcal{L}^1, \mathcal{L}^{-1}]$ holds by Lemma 6.2 (iv).

By the previous lemma, the Lie algebra $\mathcal{L}$ is isomorphic to $\mathfrak{sl}(2, J(\alpha))$. It follows from the definition that $\operatorname{Supp} J$ generates Λ' and that $\Lambda = \Lambda' + \mathbb{Z}\alpha$. It is clear that J is a simple graded Jordan algebra. Since the gradation of $\mathfrak{sl}(2, J(\alpha))$ is multiplicity free, the four subsets $\operatorname{Supp} J(\alpha)$, $\pm \alpha + \operatorname{Supp} J(\alpha)$ and $\operatorname{Supp} \operatorname{Inn} J(\alpha)$ are disjoint. In particular $J(\alpha)$ is of the class $\mathcal{J}(\Lambda')$ and $(J(\alpha), \Lambda', \alpha)$ is an admissible datum. □

Lemma 8.3. *(Main Lemma for type 1 integrable Lie algebras)*
Let $\mathcal{L} \in \mathcal{G}'$ be integrable of type 1. Then any $\beta \in \operatorname{Supp} \mathcal{L}$ with $l(\beta) \neq 0$ belongs to Σ.

Proof. Fix $\alpha \in \Sigma_1$. By the previous lemma, $\mathcal{L}$ is isomorphic to $\mathfrak{sl}(2, J(\alpha))$.

Since $l(\beta) \neq 0$, we have $l(\beta) = \pm 1$. Without loss of generality, it can be assumed that $l(\beta) = 1$. Under the previous isomorphism, L_β is identified with $e \otimes X$, where X is a homogeneous element of $J(\alpha)$. By Lemma 7.4, X is strongly invertible. Let Y be its inverse. Up to a scalar multiple, $L_{-\beta}$ is identified with $f \otimes Y$. Since $[e \otimes X, f \otimes Y] = h$, it follows that $[L_\beta, L_{-\beta}] \neq 0$, and so β belongs to Σ. □

27

9 Connection between the Centroid and the Weyl group

In this section, $\mathcal{L} \in \mathcal{G}'(\Lambda)$ is an integrable simple graded Lie algebra of type N, where $N = 1$ or $N = 2$. The Main Lemma is now established for integrable Lie algebras of both types 1 and 2. The goal of the section is to prove that $C(\mathcal{L})$ is very large, namely $\mathcal{L}$ is finitely generated as a $C(\mathcal{L})$-module.

Recall the decomposition $\mathcal{L} = \oplus_{i \in \mathbb{Z}} \mathcal{L}^i$, where $\mathcal{L}^i = \mathrm{Ker}\,(\mathrm{ad}\,L_0 - i)$.

Lemma 9.1. *There is a natural algebra isomorphism*
$$C(\mathcal{L}) \simeq \mathrm{End}_{\mathcal{L}^0}(\mathcal{L}^N).$$

Proof. Any $\psi \in C(\mathcal{L})$ commutes with L_0, and therefore ψ stabilizes $\mathcal{L}^N$. This induces a natural algebra morphism $\theta : C(\mathcal{L}) \to \mathrm{End}_{\mathcal{L}^0}(\mathcal{L}^N)$. By Lemma 6.2 (iv), $\mathcal{L}^N$ generates the adjoint module, so θ is injective.

Set $V = \mathrm{Ind}\,_{\mathcal{L}^{\geq 0}}^{\mathcal{L}}\, \mathcal{L}^N$, where $\mathcal{L}^{\geq 0} = \oplus_{i \geq 0}\, \mathcal{L}^i$. Since $\mathcal{L}^N$ generates the adjoint module, the natural $\mathcal{L}$-equivariant map $\eta : V \to \mathcal{L}$ is onto.

The $\mathcal{L}$-module V is a weakly $\Lambda \times \mathbb{Z}$-graded $\mathcal{L}$-module. In particular, there is a decomposition $V = \oplus_{i \leq N} V^i$, where $V^i = \{v \in V | L_0 v = iv\}$. Let K be the biggest $\mathcal{L}$-module lying in $V^{<N}$, where $V^{<N} = \oplus_{i<N} V^i$. It is clear that K is graded relative to the $\Lambda \times \mathbb{Z}$ gradation. Since $V^N \simeq \mathcal{L}^N$ and $\mathcal{L}$ is simple graded, it is clear that K is precisely the kernel of η.

Since $\mathrm{End}_{\mathcal{L}^0}(\mathcal{L}^N) = \mathrm{End}_{\mathcal{L}^{\geq 0}}(\mathcal{L}^N)$, it follows that any $\psi \in \mathrm{End}_{\mathcal{L}^0}(\mathcal{L}^N)$ extends to a $\mathcal{L}$-endomorphism $\hat{\psi}$ of V. Moreover $\hat{\psi}$ stabilizes each V^i, therefore it stabilizes K. Hence $\hat{\psi}$ determines an $\mathcal{L}$-endomorphism $\overline{\psi}$ of $\mathcal{L}$. Since $\overline{\psi} \in \mathrm{End}_{\mathcal{L}}(\mathcal{L}) = C(\mathcal{L})$ extends $\psi \in \mathrm{End}_{\mathcal{L}^0}(\mathcal{L}^N)$, the natural algebra morphism $\theta : C(\mathcal{L}) \to \mathrm{End}_{\mathcal{L}^0}(\mathcal{L}^N)$ is onto.

Therefore θ is an isomorphism, which proves the lemma. $\qquad\square$

From now on, denote by Λ' the subgroup of Λ generated by $\alpha - \beta$ when α, β run over Σ_1.

Lemma 9.2. *Let $\alpha \in \Sigma_1$. We have $\mathrm{Supp}\,\mathcal{L}^i \subset i\alpha + \Lambda'$ for all i.*

Proof. By the Main Lemmas 6.4 and 8.3, we have $\mathrm{Supp}\,\mathcal{L}^1 = \Sigma_1$, thus we have $\mathrm{Supp}\,\mathcal{L}^1 \subset \alpha + \Lambda'$. Similarly, we have $\mathrm{Supp}\,\mathcal{L}^{-1} \subset -\alpha + \Lambda'$. By Assertion (iv) of Lemma 6.2, we have $\mathcal{L}^0 = [\mathcal{L}^{-1}, \mathcal{L}^1]$, $\mathcal{L}^2 = [\mathcal{L}^1, \mathcal{L}^1]$ and $\mathcal{L}^{-2} = [\mathcal{L}^{-1}, \mathcal{L}^{-1}]$. Thus the inclusion $\mathrm{Supp}\,\mathcal{L}^i \subset i\alpha + \Lambda'$ easily follows. $\qquad\square$

Let $\beta \in \Sigma_1$. Since $[L_\beta, L_{-\beta}] \neq 0$, it can be assumed that $[L_\beta, L_{-\beta}] = 2L_0$. Let s_β be the automorphism of the Lie algebra $\mathcal{L}$ defined by
$$s_\beta = \exp(-\mathrm{ad}\,(L_\beta)) \circ \exp(\mathrm{ad}\,(L_{-\beta})) \circ \exp(-\mathrm{ad}\,(L_\beta)).$$

Lemma 9.3. *Let $\beta \in \Sigma_1$ and let $\lambda \in \mathrm{Supp}\,\mathcal{L}$.*

(i) *We have $s_\beta\,\mathcal{L}_\lambda \subset \mathcal{L}_{\lambda - 2l(\lambda)\beta}$.*

(ii) *If $l(\lambda) = 0$, then $s_\beta(L_\lambda) = \pm L_\lambda$.*

Proof. Set $\gamma = \lambda - l(\lambda)\beta$. The action of $\mathfrak{s}(\beta) := \mathbb{C}L_{-\beta} \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_\beta \simeq \mathfrak{sl}(2)$ on the spherical module $\mathcal{M}(\beta, \gamma)$ integrates to an action of the group $PSL(2, \mathbb{C})$, and s_β is the action of the group element $\pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, from which both assertions follow. $\qquad\square$

For $\alpha, \beta \in \Sigma_1$, set $t_{\alpha,\beta} = s_\alpha \circ s_\beta \circ s_\alpha \circ s_\beta$.

Lemma 9.4. *We have*

(i) $t_{\alpha,\beta}\, \mathcal{L}_\lambda \subset \mathcal{L}_{\lambda+4l(\lambda)(\alpha-\beta)}$, *for any* $\lambda \in \operatorname{Supp} \mathcal{L}$.

(ii) $t_{\alpha,\beta}(x) = x$ *for any* $x \in \mathcal{L}^0$.

Proof. The first point follows from the previous lemma . Let $\lambda \in \operatorname{Supp} \mathcal{L}^0$. By the previous lemma, there are $\epsilon_\alpha, \epsilon_\beta \in \{\pm 1\}$ such that $s_\alpha(L_\lambda) = \epsilon_\alpha L_\lambda$ and $s_\beta(L_\lambda) = \epsilon_\beta L_\lambda$. Therefore $t_{\alpha,\beta}(L_\lambda) = \epsilon_\alpha^2 \epsilon_\beta^2 L_\lambda = L_\lambda$. Thus $t_{\alpha,\beta}$ acts trivially on $\mathcal{L}^0$. $\qquad\square$

Lemma 9.5. *Set* $M = \operatorname{Supp} C(\mathcal{L})$. *Then we have* $8\Lambda' \subset M \subset \Lambda'$.

Proof. By Lemma 9.2, the support of each $\mathcal{L}^i$ is contained in one Λ'-coset. It follows easily that $M \subset \Lambda'$.

Let $\alpha, \beta \in \Sigma_1$, and let t be the restriction of $t_{\alpha,\beta}$ to $\mathcal{L}^N$. It follows from the previous lemma that t is an $\mathcal{L}^0$-morphism of $\mathcal{L}^N$ of degree $4N(\alpha - \beta)$. By Lemma 9.1, there exists a morphism $\psi \in C(\mathcal{L})$ whose restriction to $\mathcal{L}^N$ is t. Thus $4N(\alpha - \beta)$ belongs to M. Since $N = 1$ or 2, it is always true that $8(\alpha - \beta)$ belongs to M. Thus we get $8\Lambda' \subset M$. $\qquad\square$

Remark. *If $\mathcal{L}$ is of type 1, there is a simpler way to derive Lemma 9.5. Indeed, Lemma 7.5 easily implies that M contains $2\Lambda'$.*

10 Classification of integrable Lie algebras of the class $\mathcal{G}$

Recall that Λ is a lattice. We first state a classification result for the integrable Lie algebras of the class $\mathcal{G}'$.

Theorem 2′ *Let $\mathcal{L}$ be an integrable Lie algebra in the class $\mathcal{G}'(\Lambda)$ and let M be the support of $C(\mathcal{L})$. There exists a Lie algebra $\mathfrak{g} \in \mathcal{G}'(\Lambda/M)$ such that:*

(i) $\mathfrak{g}$ *is a simple finite dimensional Lie algebra,*

(ii) $\mathcal{L} \simeq \pi^*\mathfrak{g}$ *as a Λ-graded Lie algebra,*
    where π is the natural map $\Lambda \to \Lambda/M$.

Proof. By Lemma 2.5, there exists a simple Lie algebra $\mathfrak{g}$, and a Λ/M gradation $\mathfrak{g} = \oplus_\mu \mathfrak{g}_\mu$ of $\mathfrak{g}$ such that $\mathcal{L} \simeq \pi^*\mathfrak{g}$ as a Λ-graded algebra. Since $\mathcal{L} \in \mathcal{G}'(\Lambda)$, it is clear that $\mathfrak{g}$ belongs to $\mathcal{G}'(\Lambda/M)$.

It remains to prove that $\mathfrak{g}$ is finite dimensional. By Lemma 9.2, $\mathrm{Supp}\,\mathcal{L}$ lies in a most five Λ'-cosets. Therefore, $\mathrm{Supp}\,\mathfrak{g}$ lies in at most five Λ'/M-cosets. By Lemma 9.5, Λ'/M is finite. Therefore $\mathrm{Supp}\,\mathfrak{g}$ is finite, which implies that $\mathfrak{g}$ is finite dimensional. $\qquad\square$

By the previous theorem, the classification of all integrable Lie algebras in the class $\mathcal{G}'$ follows from the classification of finite dimensional simple Lie algebras of the class $\mathcal{G}'$.

For the class $\mathcal{G}$, the classification will be explicit. Let $\mathcal{L} = \pi^*\mathfrak{g}$ as in Theorem 2'.

Lemma 10.1. *Assume moreover that $\mathcal{L}$ belongs to the class $\mathcal{G}$. Then we have:*

(i) $\dim \mathfrak{g}_\mu = 1$ *for any* $\mu \in \Lambda/M$,

(ii) $\dim \mathfrak{g} = a2^n$ *for some* $a \in \{1,\, 3,\, 5\}$ *and some* $n \geq 0$.

Proof. It is obvious that $\dim \mathfrak{g}_\mu = 1$ for any $\mu \in \Lambda/M$. Since $\Lambda = \mathrm{Supp}\,\mathcal{L}$, it follows from Lemma 9.2 that Λ is an union of at most five Λ'-cosets, therefore the index $[\Lambda : \Lambda']$ is ≤ 5. By Lemma 9.5, M contains $8\Lambda'$, so $[\Lambda' : M]$ is a power of 2. Thus the index $[\Lambda : M]$ can be written as $a'2^{n'}$, where $a' \leq 5$. Thus this index is of the form $a2^n$, with $a = 1$, 3 or 5. Therefore
$$\dim \mathfrak{g} = [\Lambda : M] = a2^n,$$
with $a = 1$, 3 or 5. $\qquad\square$

Lemma 10.2. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of dimension $a2^n$ for some $a \in \{1,\, 3,\, 5\}$ and some $n \geq 0$. Then $\mathfrak{g}$ is of type A, or $\mathfrak{g}$ is isomorphic to $\mathfrak{sp}(4) \simeq \mathfrak{so}(5)$.*

Remark. *More precisely, a type A Lie algebra of dimension $a2^n$ for some $a \in \{1,\, 3,\, 5\}$ and some $n \geq 0$ is isomorphic to $\mathfrak{sl}(l)$ for $l = 2, 3, 4, 5, 7$ or 9. However this remark is not essential.*

Proof. Assume that $\mathfrak{g}$ is not of type A.

The dimension of the exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8 are respectively $14 = 2.7$, $52 = 4.13$, $78 = 6.13$, $133 = 7.19$ and $248 = 8.31$, and therefore $\mathfrak{g}$ is not exceptional.

Thus $\mathfrak{g}$ is of type B, C or D, and its dimension is $m(m+1)/2$ for some integer m. Consider the equation $m(m+1)/2 = a2^n$, for some $n \geq 0$ and some $a \in \{1,\, 3,\, 5\}$. Obviously $m = 1$ is a solution. For $m > 1$, m or $m+1$ is a odd factor of $m(m+1)/2$, and therefore m or $m+1$ should be 3 or 5. The case $m = 5$ being not a solution, the only solutions are: $m = 1, m = 2, m = 3$ and $m = 4$.

However the following values should be excluded:

(i) $m = 1$ because $\mathfrak{so}(2)$ is abelian,

(ii) $m = 2$ because $\mathfrak{so}(3) \simeq \mathfrak{sp}(2)$ is of type A,

(iii) $m = 3$ because $\mathfrak{so}(4)$ is not simple.

The remaining case $m = 4$ is precisely the dimension of $\mathfrak{sp}(4) \simeq \mathfrak{so}(5)$. $\qquad\square$

Let $\mathfrak{g}$ be a simple Lie algebra and let F be an abelian group. A gradation $\mathfrak{g} = \oplus_{\alpha \in F} \mathfrak{g}_\alpha$ of $\mathfrak{g}$ is called *simple* if $\dim \mathfrak{g}_\alpha = 1$ for any $\alpha \in F$. This implies that F is finite, of order $\dim \mathfrak{g}$.

Lemma 10.3. *(i) The Lie algebra $\mathfrak{sp}(4) \simeq \mathfrak{so}(5)$ does not admit a simple gradation.*

(ii) The Lie algebra $\mathfrak{sl}(n)$ does not admit admit a simple gradation for $n > 3$.

Proof. Point (i): Set $\mathfrak{g} = \mathfrak{sp}(4)$ and let $\mathfrak{g} = \oplus_{\alpha \in F} \mathfrak{g}_\alpha$ be a simple gradation of $\mathfrak{g}$. Since $\dim \mathfrak{g} = 10$, the group F is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. This gradation induces a $\mathbb{Z}/2\mathbb{Z}$-gradation $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where each component $\mathfrak{g}_i$ has dimension 5. Since $\mathfrak{g}_0$ is reductive of dimension 5, it is isomorphic with $\mathfrak{sl}(2) \oplus \mathbb{C}^2$ or $\mathbb{C}^5$ and its rank is ≥ 3. Since the rank of $\mathfrak{sp}(4)$ is two, this is impossible.

Point (ii): Set $\mathfrak{g} = \mathfrak{sl}(n)$, let $\mathfrak{g} = \oplus_{\alpha \in F} \mathfrak{g}_\alpha$ be a simple gradation of $\mathfrak{g}$ by a finite abelian group F. Let $X = \mathrm{Hom}(F, \mathbb{C}^*)$ be its character group. The gradation induces a natural action of X on $\mathfrak{g}$: an element $\chi \in X$ acts on $\mathfrak{g}_\alpha$ by multiplication by $\chi(\alpha)$.

Let $\rho : X \to \mathrm{Aut}\,(\mathfrak{g})$ be the corresponding morphism. Since the adjoint group $PSL(n)$ has index ≤ 2 in $\mathrm{Aut}\,(\mathfrak{g})$, there is a subgroup Y of X of index ≤ 2 such that $\rho(Y) \subset PSL(n)$. Let $\psi : Y \to PSL(n)$ be the corresponding morphism. The group F has order $dim\mathfrak{g} = n^2 - 1$, hence the order of Y is prime to n. Thus the map ψ can be lifted to a morphism $\hat\psi : Y \to SL(n)$.

Let K be the commutant of $\hat\psi(Y)$ and let $\mathfrak{k}$ be its Lie algebra. Let $F' = \{\alpha \in F \,|\, \chi(\alpha) = 1 \,\forall \chi \in Y\}$. Since $[X : Y] \leq 2$, the group F' has at most two elements. It is clear that $\mathfrak{k}$ is the subalgebra of fixed points under Y, hence $\mathfrak{k} = \oplus_{\beta \in F'} \mathfrak{g}_\beta$, from which it follows that
$$\dim K = \dim \mathfrak{k} \leq 2 \ .$$
However Y being commutative, $\hat\psi(Y)$ lies inside a maximal torus, and thus we have
$$\dim K \geq n - 1.$$
It follows that $n \leq 3$. $\qquad\square$

The Lie algebra $\mathfrak{sl}(2)$ has a simple $\mathbb{Z}/3\mathbb{Z}$-gradation Γ_3 defined as follows
$$\deg e = 1, \ \deg h = 0 \ \text{and} \ \deg f = -1,$$
where e, f, h is the standard basis.

The Lie algebra $\mathfrak{sl}(3)$ has a simple $\mathbb{Z}/8\mathbb{Z}$-gradation Γ_8 defined as follows:
$$\deg f_1 + f_2 = -1, \ \deg h_1 + h_2 = 0, \ \deg e_1 + e_2 = 1, \ \deg[f_1, f_2] = 2,$$
$$\deg f_1 - f_2 = 3, \ \deg h_1 - h_2 = 4, \ \deg e_1 - e_2 = 5, \ \text{and} \ \deg[e_1, e_2] = 6,$$
where $e_1, e_2, h_1, h_2, f_1, f_2$ are the standard Chevalley generators of $\mathfrak{sl}(3)$.

Lemma 10.4. *Any simple finite dimensional Lie algebra with a simple gradation is isomorphic to $(\mathfrak{sl}(2), \Gamma_3)$ or $(\mathfrak{sl}(3), \Gamma_8)$.*

31

Proof. By Lemma 10.3, it is enough to prove that Γ_3 is the unique simple gradation of $\mathfrak{sl}(2)$ and that Γ_8 is the unique simple gradation of $\mathfrak{sl}(3)$, up to isomorphism. The first assertion is clear.

Set $\mathfrak{g} = \mathfrak{sl}(3)$ and let $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ be a simple gradation of $\mathfrak{g}$ by a group Γ of order 8. Since $\mathfrak{g} \in \mathcal{G}(\Gamma)$, we have $\Sigma_1 \neq \emptyset$ by Lemma 6.1. Let $\alpha \in \Sigma_1$ and $V \cong \mathbb{C}^3$ be the natural representation of $\mathfrak{g}$. Since the $\mathfrak{s}(\alpha)$-module $\mathfrak{g}$ is spherical, it follows that $\mathfrak{s}(\alpha)$ is a principal $\mathfrak{sl}(2)$ subalgebra. So we can assume that $L_0 = h_1 + h_2$, $L_\alpha = e_1 + e_2$ and $L_{-\alpha} = f_1 + f_2$. Since $\mathfrak{g}^{-2} = \mathbb{C}[f_1, f_2]$ is one dimensional, $[f_1, f_2]$ is Γ-homogeneous. Let β be its degree.

So up to a scalar multiple, we have $L_\beta = [f_1, f_2]$, $L_{\beta+\alpha} = f_1 - f_2$, $L_{\beta+2\alpha} = h_1 - h_2$, $L_{\beta+3\alpha} = e_1 - e_2$ and $L_{\beta+4\alpha} = [e_1, e_2]$. Since L_β and $L_{\beta+4\alpha}$ are not proportional, we have $4\alpha \neq 0$. Thus Γ is cyclic and it is generated by α. Since the 8 elements of Γ are 0, $\pm\alpha$ and $\beta + i\alpha$ for $0 \leq i \leq 4$, it follows easily that $\beta = 2\alpha$. Thus the gradation is isomorphic to Γ_8. $\qquad\square$

Remark. *Of course, Lemma 10.4 can be proved by general arguments on simple finite dimensional Lie algebras. The only interest in our proof, using the class $\mathcal{G}(\Gamma)$ and Lemma 6.1, is that we do not need new notations.*
Also note that Lemma 10.4 can be seen as a particular case of the general results of the recent paper [E].

Theorem 2. *Let Λ be a lattice and let $\mathcal{L}$ be an integrable primitive Lie algebra of the class $\mathcal{G}$. Then $\Lambda = \mathbb{Z}$ and $\mathcal{L}$ is isomorphic to $A_1^{(1)}$ or $A_2^{(2)}$.*

Proof. By Theorem 2', there exists a finite abelian group F, a simple Lie algebra $\mathfrak{g}$ with a simple F-gradation and a surjective morphism $\pi : \Lambda \to F$ such that $\mathcal{L} \simeq \pi^*\mathfrak{g}$. By the previous lemma, F is cyclic. Since $\mathcal{L}$ is primitive, $\operatorname{Ker} \pi$ contains no primitive vectors. Hence $\Lambda = \mathbb{Z}$.

Moreover the graded simple Lie algebra $\mathfrak{g}$ is isomorphic to $(\mathfrak{sl}(2), \Gamma_3)$ or $(\mathfrak{sl}(3), \Gamma_8)$. It is clear that $\pi^*((\mathfrak{sl}(2), \Gamma_3)) \simeq A_1^{(1)}$, for any surjective morphism $\pi : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ and $\pi^*((\mathfrak{sl}(3), \Gamma_8)) = A_2^{(2)}$ for any surjective morphism $\pi : \mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$. It follows that $\mathcal{L}$ is isomorphic to $A_1^{(1)}$ or $A_2^{(2)}$. $\qquad\square$

Chapter III
Classification of non-integrable Lie algebras in $\mathcal{G}$

11 Rank 1 subalgebras

From now on, let Λ be a lattice and let $\mathcal{L} \in \mathcal{G}(\Lambda)$ be a non-integrable Lie algebra. The goal of this section is Lemma 11.4, namely that any graded subalgebra isomorphic to $\mathfrak{sl}(2)$ lies in a Witt algebra.

For any $\lambda \in \Lambda$, set

$$\Omega(\lambda) = \operatorname{Supp}[\mathcal{L}, L_\lambda].$$

Lemma 11.1. *Let $\lambda \in \Lambda$. Then we have $\Lambda = F + \Omega(\lambda)$, for some finite subset F of Λ.*

Proof. By Theorem 1, the function l is additive, and moreover $l \not\equiv 0$ (otherwise L_0 would be central). Since $\Omega(0) = \{\mu \in \Lambda \mid l(\mu) \neq 0\}$, we have $\Lambda = \Omega(0) \cup \alpha + \Omega(0)$, where α is any element with $l(\alpha) \neq 0$.

However by Lemma 1.4, we have $\Omega(\lambda) \equiv \Omega(0)$. Therefore we have $\Lambda = F + \Omega(\lambda)$ for some finite subset F of Λ. $\qquad\square$

For $\alpha \in \Lambda \setminus \{0\}$, set $\mathcal{L}^{(L_\alpha)} = \{x \in \mathcal{L} \mid \operatorname{ad}^n(L_\alpha)(x) = 0, \, \forall \, n >> 0\}$.

Lemma 11.2. *Let $\alpha \in \Sigma$. There are no $\lambda \in \Lambda$ such that $[\mathcal{L}, L_\lambda] \subset \mathcal{L}^{(L_\alpha)}$.*

Proof. Step 1: A subset $X \subset \Lambda$ is called α-*bounded* if for any $\beta \in \Lambda$ there exists $n(\beta) \in \mathbb{Z}$ such that $\beta + n\alpha \notin X$ for any $n \geq n(\beta)$.

Set $\mathfrak{s}(\alpha) = \mathbb{C}L_{-\alpha} \oplus \mathbb{C}L_0 \oplus \mathbb{C}L_\alpha$. By hypothesis, $\mathfrak{s}(\alpha)$ is isomorphic to $\mathfrak{sl}(2)$. For any $\beta \in \Lambda$, set $\mathcal{M}(\beta) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{\beta+n\alpha}$. As a $\mathfrak{sl}(2)$-module, $\mathcal{M}(\beta)$ is a weight module with weight multiplicities 1. It follows from Gabriel's classification ([D], 7.8.16) that $[L_\alpha, L_{\beta+n\alpha}] \neq 0$ for $n >> 0$. Therefore $\operatorname{Supp}\mathcal{L}^{(L_\alpha)}$ is α-bounded.

Step 2: Since $\operatorname{Supp}\mathcal{L}^{(L_\alpha)}$ is α-bounded, there is no finite subset F of Λ such that $F + \operatorname{Supp}\mathcal{L}^{(L_\alpha)} = \Lambda$. Therefore by Lemma 11.1, we have:
$$\Omega(\lambda) \not\subset \mathcal{L}^{(L_\alpha)},$$
which proves the lemma. $\qquad\square$

The Witt algebra $W = \operatorname{Der}\mathbb{C}[z, z^{-1}]$ has basis $L_n = z^{n+1}\dfrac{\mathrm{d}}{\mathrm{d}z}$, where n runs over $\mathbb{Z}$. We have $[L_n, L_m] = (m-n)L_{n+m}$. It contains two subalgebras $W^\pm = \operatorname{Der}\mathbb{C}[z^{\pm 1}]$. The Lie algebra W^+ has basis $(L_n)_{n \geq -1}$ and W^- has basis $(L_n)_{n \leq 1}$. Their intersection $W^+ \cap W^-$ is isomorphic to $\mathfrak{sl}(2)$.

Set $V^\pm = W/W^\pm$. Then $V^\pm$ is a $W^\pm$-module, $V^+ \oplus V^-$ is a $\mathfrak{sl}(2)$-module. Identify the elements $L_n \in W$ with their images in $V^\pm$. Then $(L_n)_{n \leq -2}$ is a basis of V^+, and $(L_n)_{n \geq 2}$ is a basis of V^-.

Let $\mathcal{V}$ be the class of all Lie algebras $\mathcal{D}$ with a basis $(\mathrm{d}_n)_{n \in \mathbb{Z}}$ satisfying $[\mathrm{d}_0, \mathrm{d}_n] = n\,\mathrm{d}_n$, for all $n \in \mathbb{Z}$. An algebra $\mathcal{D} \in \mathcal{V}$ admits a $\mathbb{Z}$-gradation, relatively to which d_n is homogeneous of degree n.

Lemma 11.3 ([M1]). *Let $\mathcal{D} \in \mathcal{V}$. Assume that $[d_1, d_{-1}] \neq 0$. As a $\mathbb{Z}$-graded Lie algebra, $\mathcal{D}$ is isomorphic to one of the following four Lie algebras:*

(i) W,

(ii) $W^+ \ltimes V^+$,

(iii) $W^- \ltimes V^-$,

(iv) $\mathfrak{sl}(2) \ltimes (V^+ \oplus V^-)$,

where, $V^\pm$ and $V^+ \oplus V^-$ are abelian ideals.

Proof. See [M1], Lemma 16. In *loc. cit.*, the statement is slightly more general, because it is only assumed that $\dim \mathcal{D}_n \leq 1$. The assumption $\dim \mathcal{D}_n = 1$ for all n corresponds with the following four types of the Lemma 16 of [M1]: type (2,2), type (2,3) (with $q = 1$), type (3,2) (with $p = 1$) and type (3,3) (with $p = q = 1$). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

For any $\alpha \in \Lambda$, set $\mathcal{L}(\alpha) = \oplus_{n\in\mathbb{Z}} \mathcal{L}_{n\alpha}$. If $l(\alpha) \neq 0$, $\mathcal{L}(\alpha)$ belongs to the class $\mathcal{V}$.

Lemma 11.4. *Let $\alpha \in \Sigma$. Then $\mathcal{L}(\alpha)$ is isomorphic to W.*

Proof. Step 1: Assume otherwise. By the previous lemma, $\mathcal{L}(\alpha)$ contains an abelian ideal. Exchanging the role of $\pm\alpha$ if necessary, we can assume that $M = \oplus_{n\leq-2} \mathcal{L}_{n\alpha}$ is an abelian ideal of $\mathcal{L}(\alpha)$.

Set $\mathcal{Q} = \oplus_{n\leq 1} \mathcal{L}_{n\alpha}$. It follows that $\mathcal{Q} \simeq \mathfrak{sl}(2) \ltimes M$. For any $\mathbb{Z}\alpha$-coset $\beta \subset \Lambda$, set $\mathcal{M}(\beta) = \oplus_{\gamma\in\beta} \mathcal{L}_\gamma$ and $\mathcal{F}(\beta) = \mathcal{M}(\beta)/\mathcal{M}(\beta)^{(L_\alpha)}$, where $\mathcal{M}(\beta)^{(L_\alpha)} = \mathcal{M}(\beta) \cap \mathcal{L}^{(L_\alpha)}$. Since $\mathrm{ad}\,(L_\alpha)$ is locally nilpotent on $\mathcal{Q}$, $\mathcal{M}(\beta)^{(L_\alpha)}$ is a $\mathcal{Q}$-submodule and thus $\mathcal{F}(\beta)$ is a $\mathcal{Q}$-module.

In the next two points, we will prove the following assertion:

$(\mathcal{A})$ for any $\mathbb{Z}\alpha$-coset β, $L_{-2\alpha}$ acts trivially on $\mathcal{F}(\beta)$.

Step 2: Assume first that $\mathcal{M}(\beta)^{(L_\alpha)} \neq 0$ and prove Assertion $(\mathcal{A})$ in this case.

It follows from $\mathfrak{sl}(2)$-theory that $\mathrm{Ker}\,\mathrm{ad}\,(L_\alpha)|_{\mathcal{M}(\beta)}$ has dimension ≤ 2 (see [D] 7.8.16 or [Mi]). So there exists $\gamma \in \beta$ such that $\mathcal{M}(\beta)^{(L_\alpha)} = \oplus_{n>0}\mathcal{L}_{\gamma-n\alpha}$. Thus, $\mathcal{F}(\beta)$ is the free $\mathbb{C}[L_\alpha]$-module of rank one generated by L_γ. Since $\mathcal{L}_{\gamma-2\alpha} \subset \mathcal{M}(\beta)^{(L_\alpha)}$, we have:
$$[L_{-2\alpha}, L_\gamma] = 0 \text{ modulo } \mathcal{M}(\beta)^{(L_\alpha)}.$$
So $L_{-2\alpha}$ acts trivially on the generator L_γ of the $\mathbb{C}[L_\alpha]$-module $\mathcal{F}(\beta)$. Since $[L_{-2\alpha}, L_\alpha] = 0$, it follows that $L_{-2\alpha}$ acts trivially on $\mathcal{F}(\beta)$.

Step 3: Assume now that $\mathcal{M}(\beta)^{(L_\alpha)} = 0$ and prove Assertion $(\mathcal{A})$ in this case.

After a renormalization of L_0, it can be assumed that $l(\alpha) = 1$. Also M is isomorphic to an irreducible Verma module, so we have $[L_\alpha, L_{-3\alpha}] \neq 0$. After a suitable renormalization, it can be assumed that $[L_{-3\alpha}, L_\alpha] = L_{-2\alpha}$.

Let t be the action of L_α on $\mathcal{M}(\beta)$ and let $\gamma \in \beta$. By hypothesis, t acts injectively on $\mathcal{M}(\beta)$. Since $t.\mathcal{M}(\beta)_{\gamma+n\alpha} \subset \mathcal{M}(\beta)_{\gamma+(n+1)\alpha}$ and $\dim \mathcal{M}(\beta)_{\gamma+n\alpha} = 1$ for any n, it follows that t acts bijectively. Hence $\mathcal{M}(\beta)$ is the free $\mathbb{C}[t, t^{-1}]$-module of rank 1 generated by L_γ.

Use this generator L_γ to identify $\mathcal{M}(\beta)$ with $\mathbb{C}[t, t^{-1}]$ and denote by $\rho : \mathcal{Q} \to \mathrm{End}(\mathbb{C}[t, t^{-1}])$ the corresponding action.

Set $d = \rho(L_0)$. Since $[L_0, L_\alpha] = L_\alpha$ and $[L_0, L_\gamma] = l(\beta)L_\gamma$, we get that $[d, t] = t$ and $d.1 = l(\gamma)$. Therefore $d = t\mathrm{d}/\mathrm{d}\,t + l(\gamma)$.

Set $X = \rho(L_{-2\alpha})$. Since $[L_\alpha, L_{-2\alpha}] = 0$ and $[L_0, L_{-2\alpha}] = -2L_{-2\alpha}$, we get $[t, X] = 0$ and $[t\mathrm{d}/\mathrm{d}\,t, X] = -2X$. So we have $X = at^{-2}$ for some $a \in \mathbb{C}$.

Set $Y = \rho(L_{-3\alpha})$. Since $[L_{-3\alpha}, L_\alpha] = L_{-2\alpha}$ and $[L_0, L_{-3\alpha}] = -3L_{-3\alpha}$, we get $[Y, t] = at^{-2}$ and $[t\mathrm{d}/\mathrm{d}\,t, Y] = -3Y$. So we have $Y = at^{-2}\mathrm{d}/\mathrm{d}\,t + bt^{-3}$ for some $b \in \mathbb{C}$.

34

Since M is an abelian ideal, we have $[\rho(L_{-3\alpha}), \rho(L_{-2\alpha})] = 0$, i.e.
$$[at^{-2}\mathrm{d}/\mathrm{d}\,t + bt^{-3}, at^{-2}] = 0,$$
from which it follows that $a = 0$ and $X = 0$. Therefore $L_{-2\alpha}$ acts trivially on $\mathcal{M}(\beta) = \mathcal{F}(\beta)$.

$Step\ 4:$ Assertion $(\mathcal{A})$ is equivalent to
$$[\mathcal{L}, L_{-2\alpha}] \subset \mathcal{L}^{(L_\alpha)}.$$
This contradicts Lemma 11.2. It follows that $\mathcal{L}(\alpha)$ is necessarily isomorphic to W. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

12 Symbols of twisted pseudo-differential operators on the circle

12.1 Ordinary pseudo-differential operators on the circle

Denote by A the Laurent polynomial ring $\mathbb{C}[z, z^{-1}]$. By definition, $\operatorname{Spec} A$ is called the *circle*. Let D^+ be the algebra of differential operators on the circle. An element in D^+ is a finite sum $a = \sum_{n \geq 0} a_n \partial^n$, where $a_n \in A$ and where ∂ stands for $\frac{\mathrm{d}}{\mathrm{d}\,z}$. The product of the differential operator a by a differential operator $b = \sum_{m \geq 0} b_m \partial^m$ is described by the following formula:
$$a.b = \sum_{k \geq 0} \sum_{n,\,m \geq 0} \binom{n}{k} a_n (\partial^k b_m) \partial^{n+m-k}.$$
A *pseudo-differential operator* on the circle is a formal series
$$a = \sum_{n \in \mathbb{Z}} a_n \partial^n,$$
where $a_n \in A$ and $a_n = 0$ for $n >> 0$. The space D of all pseudo-differential operators has a natural structure of algebra (see below for a precise definition of the product).

12.2 Twisted differential operators on the circle

It is possible to enlarge the algebra D^+ by including complex powers of z.

Let $\mathcal{A}$ be the algebra with basis $(z^s)_{s \in \mathbb{C}}$ and product $z^s z^t = z^{s+t}$. The derivation ∂ extends to $\mathcal{A}$ by $\partial z^s = s.z^{s-1}$. Set $\mathcal{D}^+ = \mathcal{A} \otimes_A D^+$. The product on D^+ extends naturally to $\mathcal{D}^+$. The algebra $\mathcal{D}^+$ will be called the *algebra of twisted differential operators* on the circle.

As usual, set $\binom{s}{k} = (1/k!)s(s-1)\ldots(s-k+1)$ for any $s \in \mathbb{C}$ and any $k \in \mathbb{Z}_{\geq 0}$. When s is a non-negative integer, $\binom{s}{k}$ is the usual binomial coefficient. The product in $\mathcal{D}^+$ is defined by the following formula:
$$z^s \partial^m . z^t \partial^n = \sum_{k \geq 0} k! \binom{m}{k}\binom{t}{k} z^{s+t-k} \partial^{m+n-k}.$$
Here the sum is finite, because $\binom{m}{k} = 0$ for $k > m$.

Let $\mathcal{D}^+_{\leq n}$ be the space of all differential operators of order $\leq n$. Set $\mathcal{P}^+ = \oplus_{n \geq 0} \mathcal{D}^+_{\leq n}/\mathcal{D}^+_{\leq n-1}$. As usual we have $\mathcal{D}^+_{\leq m}.\mathcal{D}^+_{\leq n} \subset \mathcal{D}^+_{\leq m+n}$ and $[\mathcal{D}^+_{\leq m}, \mathcal{D}^+_{\leq n}] \subset \mathcal{D}^+_{\leq m+n-1}$, therefore $\mathcal{P}^+$ has a natural structure of Poisson algebra. As usual, an element $a \in \mathcal{D}^+_{\leq n} \setminus \mathcal{D}^+_{\leq n-1}$ has exactly *order n*, and its image $\sigma(a) = a$ mod $\mathcal{D}^+_{\leq n-1}$ in $\mathcal{P}^+$ is called its *symbol*. That is why $\mathcal{P}^+$ is called the *algebra of*

symbols of twisted differential operators. In what follows, the Poisson bracket of symbols will be denoted by $\{,\}$.

12.3 Twisted pseudo-differential operators on the circle

Similarly, it is possible to enlarge the algebra D by adding complex powers of z and ∂.

Since the formula involves an infinite series in powers of ∂, a restriction on the the support of the series is necessary to ensure the convergence of the series defining the product.

For any $s \in \mathbb{C}$, set $] - \infty, s] = \{s - n \mid n \in \mathbb{Z}_{\geq 0}\}$. Say that a subset X of $\mathbb{C}$ is *good* if there exists a finite set S such that $X \subset \cup_{s \in S}] - \infty, s]$. Let $\mathcal{D}$ be the space of all formal series $\sum_x a_x \partial^x$, where $a_x \in \mathcal{A}$ for any $x \in \mathbb{C}$ and where $a_x = 0$ for all x outside a good subset of $\mathbb{C}$. An element of $\mathcal{D}$ is called a *twisted pseudo-differential operator*. Then one can define a product on $\mathcal{D}$ by the formula

$$z^s \partial^x . z^t \partial^y = \sum_{k \geq 0} k! \binom{x}{k} \binom{t}{k} z^{s+t-k} \partial^{x+y-k}.$$

Thank to the restriction on the support, the product of general twisted pseudo-differential operators is well-defined.

Unfortunately, the definition of the order of a twisted pseudo-differential operator requires a non-natural choice of a total ordering $<$ on $\mathbb{C}$ (viewed as an abelian group) in a such way that its restriction to $\mathbb{Z}$ is the usual order. Then say that the operator $a \in \mathcal{D}$ has *order x* if a can be writen as $a = a_x \partial^x + \sum_{y < x} a_y \partial^y$, where $a_x \in \mathcal{A}$ is not zero. Let $\mathcal{D}_{\leq x}$ (respectively $\mathcal{D}_{<x}$) be the subspace of all twisted pseudo-differential operators of order $\leq x$ (respectively of order $< x$) and set $\mathcal{P} = \oplus \mathcal{D}_{\leq x}/\mathcal{D}_{<x}$

We have $\mathcal{D}_{\leq x}.\mathcal{D}_{\leq y} \subset \mathcal{D}_{\leq x+y}$ and $[\mathcal{D}_{\leq x}, \mathcal{D}_{\leq y}] \subset \mathcal{D}_{\leq x+y-1}$ and therefore $\mathcal{P}$ has a natural structure of Poisson algebra. It should be noted that any good subset of $\mathbb{C}$ has a maximal element, therefore the symbol $\sigma(a)$ of any $a \in \mathcal{D}$ is well-defined. For $\lambda = (s, t) \in \mathbb{C}^2$, set $E_\lambda = \sigma(z^{s+1} \partial^{t+1})$. Then the commutative product of $\mathcal{P}$ is given by the formula $E_\lambda . E_\mu = E_{\lambda+\mu+\rho}$ and the Poisson bracket by $\{E_\lambda, E_\mu\} = < \lambda + \rho | \mu + \rho > E_{\lambda+\mu}$, where $<,>$ is the standard symplectic form on $\mathbb{C}^2$ and $\rho = (1, 1)$. Indeed for any s, $t \in \mathbb{C}$ the symbol of $z^s \partial^t$ is independent on the choice of a total order $<$. Thus the whole Poisson structure does not depend on this non-natural choice.

12.4 Decomposition of $\mathcal{P}$ under the Witt algebra W

For any integer n, set $L_n = \sigma(z^{n+1} \partial)$. We have $\{L_n, L_m\} = (m - n)L_{n+m}$. Therefore, the Lie algebra $W = \oplus_n \mathbb{C}L_n$ is isomorphic to the Witt algebra, i.e. the derivation algebra of $\mathbb{C}[z, z^{-1}]$.

Fix $\delta \in \mathbb{C}$. Set $\Omega^\delta = \mathcal{D}_{\leq-\delta}/\mathcal{D}_{<-\delta}$. Thus W is a Lie subalgebra of Ω^{-1} and each Ω^δ is a W-module. For any x, set $u_x^\delta = \sigma(z^{x-\delta} \partial^{-\delta}) = E_{x-\delta-1, -\delta-1}$. Note that

$$\{L_n, u_x^\delta\} = (x + n\delta)u_{x+n}^\delta.$$

For any coset $s \in \mathbb{C}/\mathbb{Z}$, set

$$\Omega_s^\delta = \oplus_{x \in s} \mathbb{C} u_x^\delta.$$
There is a decomposition $\Omega^\delta = \oplus_{s \in \mathbb{C}/\mathbb{Z}} \Omega_s^\delta$, where each Ω_s^δ is a W-submodule. These W-modules Ω_s^δ are called the *tensor densities modules*.

It is clear that $\Omega_0^0 = A$ and $\Omega_0^1 = \Omega_A^1$, where Ω_A^1 is the module of Kähler differential of A. As W-module, Ω_0^0 and Ω_0^1 have length two. Indeed, set $\overline{A} = \mathbb{C}[z, z^{-1}]/\mathbb{C}$. Their composition series are described by the following exact sequences:
$$0 \to \mathbb{C} \to \Omega_0^0 \to \overline{A} \to 0 \text{ and } 0 \to \overline{A} \to \Omega_0^1 \to \mathbb{C} \to 0.$$
Otherwise, for $s \neq 0$ or for $\delta \notin \{0, 1\}$, the W module Ω_s^δ is irreducible. It should be noted that $W = \Omega_0^{-1}$.

12.5 The W-equivariant bilinear maps $P_{s,s'}^{\delta,\delta'}$ and $B_{s,s'}^{\delta,\delta'}$

Set $Par = \mathbb{C} \times \mathbb{C}/\mathbb{Z}$. There is a decomposition $\mathcal{P} = \oplus_{(\delta,s) \in Par} \Omega_s^\delta$. It follows from the explicit description of the Poisson structure on $\mathcal{P}$ that we have:
$$\Omega_s^\delta . \Omega_{s'}^{\delta'} \subset \Omega_{s+s'}^{\delta+\delta'} \text{ and } \{\Omega_s^\delta, \Omega_{s'}^{\delta'}\} \subset \Omega_{s+s'}^{\delta+\delta'+1}.$$
for any quadruple $(\delta, s), (\delta', s') \in Par$. Accordingly, we get two W-equivariant bilinear maps:
$$P_{s,s'}^{\delta,\delta'} : \Omega_s^\delta \times \Omega_{s'}^{\delta'} \to \Omega_{s+s'}^{\delta+\delta'} \text{ and } B_{s,s'}^{\delta,\delta'} : \Omega_s^\delta \times \Omega_{s'}^{\delta'} \to \Omega_{s+s'}^{\delta+\delta'+1}.$$
In what follows, these morphisms $P_{s,s'}^{\delta,\delta'}$ and $B_{s,s'}^{\delta,\delta'}$ will be called the *commutative product* and the *Poisson bracket product*. These bilinear maps are always non-zero, except the Poisson bracket for $\delta = \delta' = 0$.

12.6 The outer derivations $\log z$ and $\log \partial$

Recall the following obvious fact:

Lemma 12.1. *Let R be a Poisson algebra, and let $d \in R$ be an invertible element. Then the map $r \in R \mapsto \{d, r\}/d \in R$ is a derivation.*

In what follows, it will be convenient to use the notation $\{\log d, r\}$ for $\{d, r\}/d$. Since the ordinary bracket of operators is denoted by $[,]$ and the Poisson bracket of symbol is denoted by $\{,\}$, it will be convenient to write $z^s \partial^\delta$ for $\sigma(z^s \partial^\delta)$. In an expression like $\{z^s \partial^\delta, z^{s'} \partial^{\delta'}\}$ it is clear that the arguments are symbols of operators.

In the Poisson algebra $\mathcal{P}$, both z and ∂ are invertible. Therefore $\{\log z, \cdot\}$ and $\{\log \partial, \cdot\}$ are derivations of $\mathcal{P}$. Let $\mathcal{E} \subset \text{Der} \, \mathcal{P}$ the vector space generated by $\text{ad} \, \Omega_0^0$, $\{\log z, \cdot\}$ and $\{\log \partial, \cdot\}$. Also set $\overline{A} = A/\mathbb{C} = \Omega_0^0/\mathbb{C}$. As a vector space, it is clear that $\mathcal{E} = \overline{A} \oplus \mathbb{C}\{\log z, \cdot\} \oplus \mathbb{C}\{\log \partial, \cdot\}$.

We have:
$$\{\log z, z^s \partial^\delta\} = -\delta z^{s-1} \partial^{\delta-1}, \quad \{\log \partial, z^s \partial^\delta\} = s z^{s-1} \partial^{\delta-1}.$$
It follows that $\mathcal{E}$ is a W-module, with basis $\{\log z, \log \partial, (e_n) | \, n \neq 0\}$ and the W-module structure is given by:

$L_n . e_m = m e_{n+m}$ if $n + m \neq 0$ and 0 otherwise,

$L_n . \log z = e_n$ if $n \neq 0$ and 0 otherwise,

$L_n . \log \partial = -(n+1)e_n$ if $n \neq 0$ and 0 otherwise,

where e_n is the image of z^n in $\overline{A}$.

Lemma 12.2. *(i) As a W-module, there is an exact sequence:*
$$0 \to \overline{A} \to \mathcal{E} \to \mathbb{C}^2 \to 0$$

(ii) $\mathcal{E}.\Omega_s^\delta \subset \Omega_s^{\delta+1}$, for all $(\delta, s) \in Par$.

Proof. Both assertions follow from the previous computations. $\qquad\square$

12.7 The Lie algebra W_π

As a Lie algebra, $\mathcal{P}$ has a natural $\mathbb{C}^2$-gradation $\mathcal{P} = \oplus_{\lambda \in \mathbb{C}^2} \mathcal{P}_\lambda$ where $\mathcal{P}_\lambda = \mathbb{C}E_\lambda$. For any additive map $\pi : \Lambda \to \mathbb{C}^2$ set, $W_\pi = \pi^*\mathcal{P}$. When π is one-to-one, W_π has been defined in the introduction. In general, the notation π^* has been defined in Section 2.

Lemma 12.3. *(i) The Lie algebra W_π is simple graded iff:*
$$(\mathcal{C}) \qquad \mathrm{Im}\ \pi \not\subset \mathbb{C}\rho\ \text{and}\ 2\rho \notin \mathrm{Im}\ \pi.$$

(ii) Moreover if π is one to one, then W_π is simple.

Proof. We may assume that π is one-to-one and that Λ is a subgroup of $\mathbb{C}^2$. Thus W_π has basis $(E_\lambda)_{\lambda \in \Lambda}$ and the bracket is given by the formula
$$[E_\lambda, E_\mu] = <\lambda + \rho|\mu + \rho> E_{\lambda+\mu}.$$

If $\Lambda \subset \mathbb{C}\rho$, then W_π is abelian. If $2\rho \in \Lambda$, then $E_{2\rho} \in W_\pi \setminus [W_\pi, W_\pi]$. Thus, the condition $\mathcal{C}$ is necessary for the simplicity of W_π.

Conversely, assume the condition $\mathcal{C}$. First prove that W_π is simple as a graded Lie algebra. Let $\lambda, \mu \in \Lambda$. For $\theta \in \Lambda$, set

$g(\theta) = <\lambda + \rho|\theta + \rho>$

$h(\theta) = <\lambda + \theta + \rho|\mu + 2\rho>$

Since $2\rho \notin \Lambda$, we have $\lambda + \rho \neq 0$ and $\mu + 2\rho \neq 0$, and so g and h are not constant affine functions. Since their zero sets are proper cosets of Λ, there exists $\theta \in \Lambda$ with $g(\theta)h(\theta) \neq 0$.

Note that $h(\theta) = <\lambda + \theta + \rho|\mu - \lambda - \theta + \rho>$, and therefore:
$$[E_{\mu-\theta-\lambda}, [E_\theta, E_\lambda]] = g(\theta)h(\theta)E_\mu.$$
It follows that for any $\lambda, \mu \in \Lambda$, E_μ belongs to $\mathrm{Ad}\,(U(W_\pi))(E_\lambda)$. So W_π is simple as a Λ-graded Lie algebra.

Next prove that W_π is simple. Let ψ be an homogeneous element of the centroid $C(W_\pi)$, and let μ be its degree. Since ψ is injective and since $\psi(L_0)$ commutes with L_0, we have $\mu = z\rho$ for some $z \in \mathbb{C}$.

Fix any $\lambda \in \Lambda$ with $<\lambda|\rho> \neq 0$. Define the function $c : \Lambda \to \mathbb{C}$ by the requirement:
$$\mathrm{ad}\,(E_{-\lambda})\mathrm{ad}\,(E_\lambda)(E_\mu) = c(\mu)E_\mu.$$
Using the facts that ψ commutes with $\mathrm{ad}\,(E_{-\lambda})\mathrm{ad}\,(E_\lambda)$ and that ψ is injective, it follows that $c(\mu + z\rho) = c(\mu), \forall \mu \in \Lambda$ and therefore the function $n \in \mathbb{Z} \mapsto c(\mu + nz\rho)$ is constant. However we have $c(\mu) = <-\lambda + \rho|\lambda + \mu + \rho><\lambda + \rho|\mu + \rho>$, so $c(\mu + nz\rho)$ is a degree two polynomial in n with highest term is $-z^2 <\lambda|\rho>^2 n^2$. Therefore $z = 0$, which means that $C(W_\pi) = \mathbb{C}$.

It follows from Section 2, in particular Lemma 2.5, that W_π is simple. $\qquad\square$

13 Tensor products of generalized tensor densities modules

This section is a review of the results of [KS] and [IM] which are used later on. Indeed [IM] contain the whole list of all W-equivariant bilinear maps μ : $M^1 \times M^2 \to N$, where M^1, M^2 and N are in a certain class $\mathcal{S}(W)$ (defined below). However the classification contains many cases. Here we will only state the consequences which are of interest for this paper.

13.1 The Kaplansky Santharoubane Theorem

As before, $W = \operatorname{Der} \mathbb{C}[z, z^{-1}]$ denotes the Witt Lie algebra. Given a W-module M, set $M_x = \{m \in M | L_0.m = xm\}$. Let $\mathcal{S}(W)$ be the class of all W-modules such that there exists $s \in \mathbb{C}/\mathbb{Z}$ satisfying the following conditions:

(i) $M = \oplus_{x \in s} M_x$

(ii) $\dim M_x = 1$ for all $x \in s$.

All the tensor densities modules Ω_s^δ, which have been defined in Section 12, are in the class $\mathcal{S}(W)$. Conversely, the modules in the class $\mathcal{S}(W)$ are called *generalized tensor densities modules*. For $(a, b) \in \mathbb{C}^2$, define the modules $A_{a,b}$ and $B_{a,b}$ as follows:

(i) the module $A_{a,b}$ has basis $(u_n)_{n \in \mathbb{Z}}$ and we have

$$L_m.u_n = (m + n)u_{m+n} \quad \text{if } n \neq 0,$$
$$L_m.u_0 = (am^2 + bm)u_m.$$

(ii) the module $B_{a,b}$ has basis $(v_n)_{n \in \mathbb{Z}}$ and we have

$$L_m.v_n = nv_{m+n} \quad \text{if } m + n \neq 0,$$
$$L_m.v_{-m} = (am^2 + bm)v_0.$$

The family of modules $A_{a,b}$ with $(a, b) \in \mathbb{C}^2$ is called the *A-family*. Similarly, the family of all modules $B_{a,b}$ is called the *B-family*. The union of the two families is called the *AB-family*.

Set $A = \mathbb{C}[z, z^{-1}]$ and $\overline{A} = A/\mathbb{C}$. If M is in A-family, then there is an exact sequence $0 \to \overline{A} \to M \to \mathbb{C} \to 0$. Similarly, if N is in B-family, then there is an exact sequence $0 \to \mathbb{C} \to N \to \overline{A} \to 0$. Since we have $\overline{A} \oplus \mathbb{C} \simeq A_{0,0} \simeq B_{0,0}$, the W-module $\overline{A} \oplus \mathbb{C}$ is in both families. Otherwise the modules $A_{a,b}$ and $B_{a,b}$ are indecomposable. Since $A_{a,b} \simeq A_{xa,xb}$ (respectively $B_{a,b} \simeq B_{xa,xb}$) for any non-zero scalar x, the indecomposable modules of the A-family (respectively of the B-family) are parametrized by $\mathbb{P}^1$.

Indeed I. Kaplansky and R. Santharoubane gave a complete classification of all modules in $\mathcal{S}(W)$.

Theorem [KS]: (Kaplansky-Santharoubane Theorem)

(i) *Any irreducible module $M \in \mathcal{S}(W)$ is isomorphic to Ω_s^δ for some $\delta \in \mathbb{C}$, $s \in \mathbb{C}/\mathbb{Z}$ with the condition $s \neq 0$ if $\delta = 0$ or $\delta = 1$.*

(ii) *Any reducible module $M \in \mathcal{S}(W)$ is in the AB-family.*

The theorem is due to Kaplansky and Santharoubane. The original paper [KS] is correct, but the statement contains a little misprint (the indecomposable modules were classified by the affine line instead of the projective line). For the A-family, see [MS] for a correct statement, and in general see [MP] (see also [Kap]).

13.2   Degree of modules in $\mathcal{S}(W)$

The de Rham differential provides a W-equivariant map d $: \Omega_s^0 \to \Omega_s^1$. For $s \notin \mathbb{Z}$, the map d is an isomorphism. Otherwise, the modules Ω_s^δ are pairwise non-isomorphic. Therefore one can define *the degree* $\deg M$ of any $M \in \mathcal{S}(W)$ as follows:

   $\deg M = \{\delta\}$ if $M \simeq \Omega_s^\delta$ for some $\delta \neq 0$, 1, and
   $\deg M = \{0, 1\}$ otherwise.

Note that the degree is a multivalued function. By definition, *a degree* for M is a value $\delta \in \deg M$. Let $\mathcal{S}^*(W)$ be the class of pairs (M, δ) where $M \in \mathcal{S}(W)$ and δ is a degree of M. For example, $(\Omega_s^0, 0)$ and $(\Omega_s^0, 1)$ belong to $\mathcal{S}^*(W)$. Usually an element of $(M, \delta) \in \mathcal{S}^*(W)$ will be simply denoted by M, and we will set $\deg M = \delta$. So the degree is an ordinary function on $\mathcal{S}^*(W)$.

13.3   Degree of bilinear maps in $\mathcal{S}(W)$

Let $\mathcal{B}(W)$ (respectively $\mathcal{B}^*(W)$) be the set of all W-equivariant bilinear maps $\mu : M^1 \times M^2 \to N$, where M^1, M^2 and N are in $\mathcal{S}(W)$ (respectively in $\mathcal{S}^*(W)$). Let $\mu \in \mathcal{B}(W)$ (respectively $\mu \in \mathcal{B}^*(W)$). By definition the *degree* of μ is the set (respectively the number) $\deg \mu = \deg N - \deg M^1 - \deg M^2$. As before, the degree is a multivalued map on $\mathcal{B}(W)$ and an ordinary map on $\mathcal{B}^*(W)$.

Let now $\mu : M^1 \times M^2 \to N$ be a non-zero bilinear map, where M^1, M^2, N are in $\mathcal{S}^*(W)$. Set $\delta_1 = \deg M^1$, $\delta_2 = \deg M^2$ and $\gamma = \deg N$.

Lemma 13.1 ([IM])**.** *Let μ, δ_1, δ_2 and γ as before. Then $\deg \mu \in [-2, 3]$. Moreover the possible triples $(\delta_1, \delta_2, \gamma)$ are the following:*

(i) *if $\deg \mu = 3$: only $(-2/3, -2/3, 5/3)$, $(0, 0, 3)$, $(0, -2, 1)$ or $(-2, 0, 1)$,*

(ii) *if $\deg \mu = 2$: any triple $(0, \delta, \delta + 2)$, $(\delta, 0, \delta + 2)$ or $(\delta, -1 - \delta, 1)$,*

(iii) *if $\deg \mu = 1$: any triple $(\delta_1, \delta_2, \delta_1 + \delta_2 + 1)$,*

(iv) *if $\deg \mu = 0$: any triple $(\delta_1, \delta_2, \delta_1 + \delta_2)$,*

(v) *if $\deg \mu = -1$: any triple $(1, \delta, \delta)$, $(\delta, 1, \delta)$ or $(\delta, 1 - \delta, 0)$,*

(vi) *if $\deg \mu = -2$: only $(1, 1, 0)$.*

For the proof, see [IM], Corollary 3 in Section 11.

13.4 Bilinear maps of degree 1

Recall from Section 12.5 that the Poisson bracket of symbols induces a W-equivariant bilinear map:

$$B_{u,u'}^{\delta,\delta'} : \Omega_u^\delta \times \Omega_{u'}^{\delta'} \to \Omega_{u+u'}^{\delta+\delta'+1},$$

for any $(\delta, u), (\delta', u) \in P$.

Lemma 13.2. *Let $\mu : \Omega_u^\delta \times \Omega_{u'}^{\delta'} \to \Omega_{u+u'}^{\delta+\delta'+1}$ be a W-equivariant map.*

(i) *If $(\delta, \delta') \neq (0,0)$, then μ is proportional to $B_{u,u'}^{\delta,\delta'}$.*

(ii) *If $(\delta, \delta') = (0,0)$, there exist two scalars a,b such that*
$$\mu(f,g) = af\mathrm{d}\,g + bg\mathrm{d}\,f,\ \textit{for any } (f,g) \in \Omega_u^0 \times \Omega_{u'}^0,$$
where d *is the de Rham differential.*

Proof. By Corollary 4 of [IM], there are no degenerate maps $\pi : \Omega_u^\delta \times \Omega_{u'}^{\delta'} \to \Omega_{u+u'}^{\delta+\delta'+1}$. Thus the lemma follows from Theorems 3.1 and 3.2 of [IM]. $\qquad\square$

13.5 Degenerated bilinear maps with value in W

Let $\mu : M^1 \times M^2 \to N$ be a bilinear map in the class $\mathcal{B}(W)$. The bilinear map μ is called *non-degenerated* if the set of all (x,y) such that $\mu(M_x \times N_y) \neq 0$ is Zarisky dense in $\mathbb{C}^2$. Otherwise, it is called *degenerated*.

Lemma 13.3. *Let $\mu : M^1 \times M^2 \to W$ be a non-zero degenerated bilinear map in $\mathcal{B}(W)$. Then one of the following assertion holds:*

(i) *There is a non-zero morphism $\phi : M^1 \to \mathbb{C}$ of W-modules and an isomorphism $\psi : M^2 \to W$ of W-modules such that $\mu(m_1, m_2) = \phi(m_1)\psi(m_2)$ for all $m_1 \in M^1$, $m_2 \in M^2$.*

(ii) *There is an isomorphism $\psi : M^1 \to W$ of W-modules and a non-zero morphism $\phi : M^2 \to \mathbb{C}$ of W-modules such that $\mu(m_1, m_2) = \phi(m_2)\psi(m_1)$ for all $m_1 \in M^1$, $m_2 \in M^2$.*

Proof. By Corollary 2 of [IM], either M^1 or M^2 is reducible. So, assume that M^1 is reducible. Then, Assertion (i) follows from Lemma 13 of [IM]. $\qquad\square$

13.6 Non-degenerated bilinear maps with value in W

Two families of W-equivariant bilinear maps $\mu : M \times N \to W$ will be defined. Since $W \simeq \Omega_0^{-1}$, the product of symbols $\pi_s^\delta : \Omega_s^\delta \times \Omega_{-s}^{-1-\delta} \to W$ and the Poisson bracket of symbols $\beta_s^\delta : \Omega_s^\delta \times \Omega_{-s}^{-2-\delta} \to W$ are W-equivariant bilinear maps. With the notations of Section 12.5, we have $\pi_s^\delta = P_{s,-s}^{\delta,-1-\delta}$ and $\beta_s^\delta = B_{s,-s}^{\delta,-1-\delta}$.

The *ordinary family* is the set of all these maps π_s^δ and β_s^δ.

The second family is called the complementary family. Its definition requires the following obvious Lemma.

Lemma 13.4. *The indecomposable modules of the A-family are exactly the codimension one submodules of $\mathcal{E}$.*

The lemma follows easily from the structure constant of module $\mathcal{E}$ given in Section 12.6 and the the structure constant of the modules $A_{a,b}$ given in Section 13.1.

The left complementary family consists of bilinear maps $\mu_M^l : M \times \Omega_0^{-2} \to W$, where M belongs to the AB-family, which are defined as follows.

If M is an indecomposable module in the A-family, the Lemma 13.4 provides an injection $j : M \to \mathcal{E}$. Otherwise M contains a trivial submodule and $M/\mathbb{C} \simeq \overline{A}$. Thus there is a natural map $j : M \to \mathcal{E}$ whose image is $\overline{A}$.

Recall that $\mathrm{ad}\,(\mathcal{E})(\Omega_0^\delta) \subset \Omega_0^{\delta+1}$ and that $\Omega_0^{-1} \simeq W$. So define $\mu_M^l : M \times \Omega_0^{-2} \to W$ by $\mu_M^l(m,\omega) = \mathrm{ad}\,(j(m))(\omega)$. This family of maps μ_M^l is called the *left complementary family*. The *right complementary family* is the family of maps $\mu_M^r : \Omega_0^{-2} \times M \to W$ obtained by exchanging the two factors. The *complementary family* is the union of the left and of the right complementary families.

It should be noted that these definitions contain the following ambiguities. First, when $s \neq 0 \bmod \mathbb{Z}$, the de Rham operator d is an isomorphism $\mathrm{d} : \Omega_s^0 \to \Omega_s^1$, which provides the following commutative diagram:

$$
\begin{array}{ccc}
\Omega_s^0 \times \Omega_{-s}^{-2} & \xrightarrow{\ \beta_s^0\ } & \Omega_0^{-1} \\[4pt]
\downarrow{\scriptstyle d\times\mathrm{id}} & & \downarrow{\scriptstyle -\frac{1}{2}} \\[4pt]
\Omega_s^1 \times \Omega_{-s}^{-2} & \xrightarrow{\ \pi_s^1\ } & \Omega_0^{-1}
\end{array}
$$

So, up to conjugation, we have $\beta_s^0 = \pi_s^1$ and $\beta_s^{-2} = \pi_s^{-2}$. Next, the maps $\pi_0^1 : \Omega_0^1 \times \Omega_0^{-2} \to W$ and $\beta_0^0 : \Omega_0^0 \times \Omega_0^{-2} \to W$ are in the left complementary family. Similarly, β_0^{-2} and π_0^{-2} are in the right complementary family.

Lemma 13.5. *Any W-equivariant non-degenerated bilinear map $\mu : M \times N \to W$ belongs to the ordinary or to the complementary family. More precisely, μ is conjugated to one of the following map:*

(i) *the map π_s^δ for some $\delta \neq 1, -2$,*

(ii) *the map β_s^δ, for $(\delta, s) \neq (0,0), (-2,0)$,*

(iii) *or to a map of the complementary family.*

Proof. See Theorem 3.1. or Table 3 in [IM]. $\qquad\square$

14 The Main Lemma (non-integrable case)

Let $\mathcal{L} \in \mathcal{G}$ be a non-integrable Lie algebra. Recall that Σ is the set of all $\lambda \in \Lambda$ such that $\mathcal{L}_\lambda \oplus \mathcal{L}_0 \oplus \mathcal{L}_{-\lambda}$ is isomorphic to $\mathfrak{sl}(2)$.

The aim of this section is the proof of the Main Lemma in the non-integrable case, Lemma 14.8. A similar statement had been already proved for integrable Lie algebras of type I (see Lemma 8.3) and of type II (see Lemma 6.4). However the proof in the three cases are quite different.

Lemma 14.1. *Let M, $N \in \mathcal{S}(W)$ and let $\pi : M \times N \to W$ be a non-zero W-equivariant bilinear map. Then one of the following three assertions holds:*

(i) *$\pi(M_x \times N_{-x}) \neq 0$ for all $x \in \operatorname{Supp} M$ with $x \neq 0$.*

(ii) *There is a surjective morphism $\phi : M \to \mathbb{C}$ and an isomorphism $\psi : N \to W$ such that $\pi(m, n) = \phi(m)\psi(n)$ for all $m \in M$, $n \in N$.*

(iii) *There is an isomorphism $\phi : M \to W$ and a surjective morphism $\psi : N \to \mathbb{C}$ such that $\pi(m, n) = \psi(n)\phi(m)$ for all $m \in M$, $n \in N$.*

Proof. Assume that neither Assertion (ii) or (iii) hold. Then by Lemma 13.3, the map π is non-degenerated. Moreover, the non-degenerated bilinear maps $\pi : M \times N \to W$ are classified by Lemma 13.5. They are isomorphic to π_s^δ, β_s^δ or they are in the complementary family.

If $\pi = \pi_s^\delta$ for some $(\delta, s) \in P$, then π is the restriction of the commutative product of the Poisson algebra $\mathcal{P}$. Since the commutative algebra $\mathcal{P}$ has no zero divisors, it follows that $\pi(m, n) \neq 0$ whenever m and n are not zero, thus the assertion is clear if $\pi = \pi_s^\delta$ for some $(\delta, s) \in P$.

If $\pi = \beta_s^\delta$ for some $(\delta, s) \in P$, then π is the restriction of the Poisson bracket of $\mathcal{P}$. Let $x \in \operatorname{Supp} M$ and let $m \in M_x$, $n \in N_{-x}$ be non-zero vectors. When M and N are realized as submodules of $\mathcal{P}$, m and n are identified with vectors which are homogeneous in z and ∂. So, up to some non-zero scalar, we have $m = E_\lambda$ and $n = E_{-\lambda}$ for some $\lambda \in \mathbb{C}^2$ and $\pi(m, n)$ can be identified with $\{E_\lambda, E_{-\lambda}\}$. However we have $\{E_\lambda, E_{-\lambda}\} = 2 < \lambda | \rho > E_0$, and therefore we get:
$$\{E_\lambda, E_{-\lambda}\} = 0 \text{ implies that } \{E_0, E_\lambda\} = 0.$$
So $\pi(m, n) = 0$ implies that $x = 0$. Thus the assertion is proved if $\pi = \beta_s^\delta$ for some $(\delta, s) \in P$.

Assume now that π is in the left complementary family. Since $\Omega_0^0/\mathbb{C} \simeq \overline{A}$, the map $\beta_0^0 : \Omega_0^0 \times \Omega_0^{-2} \to W$ induces a map $\beta : \overline{A} \times \Omega_0^{-2} \to W$. It follows from the previous computation that $\beta(a, \omega) \neq 0$ if a and ω are non-zero eigenvectors of L_0 with opposite eigenvalues. Since any bilinear map of the left complementary family is a lift, or an extension of β, it follows that $\pi(M_x \times N_{-x}) \neq 0$ for any $x \neq 0$. $\square$

A subset S of Λ is called *quasi-additive* if it satisfies the following three conditions:

(i) we have $l(\lambda) \neq 0$ for any $\lambda \in S$,

(ii) $S = -S$, and

(iii) we have $\lambda + \mu \in S$ for any $\lambda, \mu \in S$ with $l(\lambda + \mu) \neq 0$.

For a quasi-additive set S, let $K(S)$ be the set of all elements of the form $\lambda + \mu$ where λ and μ belong to S and $l(\lambda + \mu) = 0$. Set $Q(S) = S \cup K(S)$.

Lemma 14.2. *Let S be a quasi-additive subset of Λ. Then $Q(S)$ and $K(S)$ are subgroups of Λ.*

Proof. It is clear that $K(S) = Q(S) \cap \operatorname{Ker} l$. So it is enough to prove that $Q(S)$ is a subgroup. Moreover $Q(S) = -Q(S)$, so it is enough to show that $Q(S)$ is stable by addition. By definition, we have $Q(S) = S \cup (S + S)$. So it is enough to show that $S + S + S \subset S + S$.

Let λ_1, λ_2, $\lambda_3 \in S$. There are two distinct indices $1 \leq i < j \leq 3$ such that $l(\lambda_i + \lambda_j) \neq 0$. Otherwise we would have $l(\lambda_1) = -l(\lambda_2) = l(\lambda_3) = -l(\lambda_1)$ which would contradict $l(\lambda_1) \neq 0$. Since all indices play the same role, we can assume that $l(\lambda_1 + \lambda_2) \neq 0$. It follows that $\lambda_1 + \lambda_2 \in S$ and therefore $\lambda_1 + \lambda_2 + \lambda_3 = (\lambda_1 + \lambda_2) + \lambda_3 \in S + S$. $\qquad\square$

As it is well-known, if $L = L_1 \cup L_2$, where L_1, L_2 and L are subgroups of Λ, then $L = L_1$ or $L = L_2$. Indeed a similar property holds for quasi-additive subsets.

Lemma 14.3. *Let S, S_1, S_2 be quasi-additive subsets of Λ. If $S = S_1 \cup S_2$, then $S = S_1$ or $S = S_2$.*

Proof. Assume otherwise. Choose $\alpha \in S \setminus S_2$ and $\beta \in S \setminus S_1$. Since $l(\beta) \neq 0$, there exists $\epsilon \in \{\pm 1\}$ with $l(\alpha) + l(\epsilon\beta) \neq 0$.

Since $\alpha + \epsilon\beta \in S$, $\alpha + \epsilon\beta$ belongs to S_1 or to S_2, it can be assumed that $\alpha + \epsilon\beta \in S_1$. However we have:
$$\epsilon\beta = (\alpha + \epsilon\beta) - \alpha, \quad \text{and} \quad l(\alpha + \epsilon\beta) + l(-\alpha) \neq 0,$$
which implies that $\epsilon\beta = -\alpha + (\alpha + \epsilon\beta)$ belongs to S_1. This contradicts the hypothesis $\beta \in S \setminus S_1$. $\qquad\square$

Let $\mathcal{L} \in \mathcal{G}$ be a non-integrable Lie algebra.

Lemma 14.4. *The set Σ is quasi-additive.*

Proof. It is obvious that Σ satisfies the conditions (i) and (ii) of the definition of quasi-additivity.

In order to prove condition (iii), consider α, $\beta \in \Sigma$ with $l(\alpha) + l(\beta) \neq 0$. Set $\mathcal{L}(\alpha) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{n\alpha}$ and $\mathcal{M}(\pm\beta) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{\pm\beta + n\alpha}$. We have
$$[\mathcal{M}(\beta), \mathcal{M}(-\beta)] \subset \mathcal{L}(\alpha),$$
therefore the Lie bracket induces a $\mathcal{L}(\alpha)$-equivariant bilinear map:
$$\pi : \mathcal{M}(\beta) \times \mathcal{M}(-\beta) \to \mathcal{L}(\alpha).$$

By Lemma 11.4, the Lie algebra $\mathcal{L}(\alpha)$ is isomorphic to W and by hypothesis $\mathcal{M}(\pm\beta)$ belong to $\mathcal{S}(W)$. Moreover we have $\pi(L_\beta, L_{-\beta}) \neq 0$. Therefore π cannot satisfy Assertion (ii) or Assertion (iii) of the Lemma 14.1. Thus Assertion (i) of Lemma 14.1 holds. Since $l(\alpha + \beta) \neq 0$, it follows that $\pi(L_{\alpha+\beta}, L_{-\alpha-\beta}) \neq 0$, and therefore $\alpha + \beta$ belongs to Σ. $\qquad\square$

Set $Q = Q(\Sigma)$.

Lemma 14.5. *Assume $Q \neq \Lambda$. Then there exists $\delta \in \Lambda \setminus Q$ such that*
$$[L_\delta, L_{-\delta}] \neq 0 \text{ and } l(\delta) = 0.$$

Proof. By Lemma 3.3, the set Π generates Λ. So there is an element $\delta \in \Pi$ with $\delta \notin Q$. Since Σ lies in Q, δ lies in $\Pi \setminus \Sigma$. Therefore, we have $[L_\delta, L_{-\delta}] \neq 0$ and $l(\delta) = 0$. $\qquad\blacksquare$

Recall that L_λ^* is the element of the graded dual $\mathcal{L}'$ defined by
$$< L_\lambda^* | L_\mu > = \delta_{\lambda,\mu},$$
where $\delta_{\lambda,\mu}$ is Kronecker's symbol. Similarly, denote by L_n^* the dual basis of the graded dual W' of W. Assume that $Q \neq \Lambda$, and let $\delta \in \Lambda \setminus Q$ be the element defined in the previous lemma.

Lemma 14.6. *For each $\alpha \in \Sigma$, there exists a sign $\epsilon = \epsilon(\alpha)$ such that*
$$L_{n\alpha}.L_{\epsilon\delta}^* = 0 \text{ and } L_{n\alpha}.L_{-\epsilon\delta}^* \neq 0$$
for all non-zero integer n.

Proof. Set $\mathcal{L}(\alpha) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{n\alpha}$ and $\mathcal{M}(\pm\delta) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{\pm\delta+n\alpha}$. We have $[\mathcal{M}(\delta), \mathcal{M}(-\delta)] \subset \mathcal{L}(\alpha)$, therefore the Lie bracket induces a $\mathcal{L}(\alpha)$-equivariant bilinear map:
$$\pi : \mathcal{M}(\delta) \times \mathcal{M}(-\delta) \to \mathcal{L}(\alpha).$$
By Lemma 11.4, the Lie algebra $\mathcal{L}(\alpha)$ is isomorphic to W, and therefore $\mathcal{M}(\delta)$ and $\mathcal{M}(-\delta)$ belong to $\mathcal{S}(W)$.

Since we have $\Sigma \subset Q$, it follows that $\delta + \alpha \notin Q$ and therefore we get $\delta + \alpha \notin \Sigma$. Since $l(\delta + \alpha) = l(\alpha) \neq 0$, it follows that $\pi(L_{\delta+\alpha}, L_{-\delta-\alpha}) = 0$. Using again that $l(\delta + \alpha) \neq 0$, π satisfies Assertion (ii) or Assertion (iii) of Lemma 14.1.

First, assume that π satisfies Assertion (ii) of Lemma 14.1. Thus there are a surjective morphism $\phi : \mathcal{M}(\delta) \to \mathbb{C}$ and an isomorphism $\psi : \mathcal{M}(-\delta) \to \mathcal{L}(\alpha)$ such that $\pi(m, n) = \phi(m)\psi(n)$ for all $(m, n) \in \mathcal{M}(\delta) \times \mathcal{M}(-\delta)$. Since ϕ is proportional to L_δ^*, it follows that L_δ^* is $\mathcal{L}(\alpha)$-invariant. In particular, we have $L_{n\alpha}.L_\delta^* = 0$ for all integer n. Moreover, ψ induces an isomorphism $\mathcal{M}(-\delta)' \simeq W'$ under which $L_{-\delta}^*$ is a non-zero multiple of L_0^*. Since $L_n.L_0^* = 2nL_{-n}^*$, we have $L_{n\alpha}.L_{-\delta}^* \neq 0$ for all non-zero integer n. Hence, if π satisfies Assertion (ii) of the Lemma, then the Lemma holds for $\epsilon = 1$.

Otherwise, π satisfies Assertion (iii), and the same proof shows that the Lemma holds for $\epsilon = -1$. $\qquad\blacksquare$

Assume again that $Q \neq \Lambda$. Let $\delta \in \Lambda \setminus Q$ be the element defined in Lemma 14.5 and let $\epsilon : \Sigma \to \{\pm1\}, \alpha \mapsto \epsilon(\alpha)$ be the function defined by the previous lemma.

Lemma 14.7. *The function $\epsilon : \Sigma \to \{\pm1\}$ is constant.*

Proof. Let $S_\pm$ be the set of all $\alpha \in \Sigma$ with $\epsilon(\alpha) = \pm1$. We claim that the sets $S_\pm$ are quasi-additive.

For simplicity, let consider S_+. Let $\alpha, \beta \in S_+$ with $l(\alpha + \beta) \neq 0$. As before, set $\mathcal{L}(\alpha) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{n\alpha}$ and $\mathcal{M}(\beta) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{\beta+n\alpha}$. Also let $\mathcal{N}$ be the $\mathcal{L}(\alpha)$-module generated by L_β. Since $\mathcal{L}(\alpha)$ is isomorphic to W, the module $\mathcal{M}(\beta)$ belongs to $\mathcal{S}(W)$. Since $[L_0, L_\beta] \neq 0$, the module $\mathcal{N}$ is not trivial. Therefore, $\mathcal{N} = \mathcal{M}(\beta)$ or $\mathcal{M}(\beta)/\mathcal{N} \simeq \mathbb{C}$. It follows that $\mathcal{N}$ contains $L_{\alpha+\beta}$.

By definition of S_+, L^*_δ is $\mathcal{L}(\alpha)$-invariant. Since it is also invariant by L_β, we have $\mathcal{N}.L^*_\delta = 0$. In particular, we have $L_{\alpha+\beta}.L^*_\delta = 0$. This implies that $\epsilon(\alpha + \beta) = 1$, i.e. $\alpha + \beta \in S_+$. So S_+ satisfies Condition (iii) of the definition of quasi-additivity. Since the other conditions are obvious, S_+ is quasi-additive.

Similarly S_- is quasi-additive. Since $\Sigma = S_+ \cup S_-$, it follows from Lemma 14.3 that $\Sigma = S_+$ or $\Sigma = S_-$. So $\epsilon(\alpha)$ is independant of α and the lemma is proved. $\qquad\square$

Lemma 14.8. *(Main Lemma non-integrable Lie algebras) Let $\mathcal{L} \in \mathcal{G}$ be a non-integrable Lie algebra. Then Σ is the set of all $\lambda \in \Lambda$ such that $l(\lambda) \neq 0$.*

Proof. Assume otherwise. By Lemma 14.2, the subgroup Q generated by Σ is proper. Let $\delta \in \Lambda \setminus Q$ be the element of Lemma 14.5. By Lemma 14.6, the function $\epsilon : \Sigma \to \{\pm 1\}$ is constant. In order to get a contradiction, we can assume that $\epsilon(\gamma) = 1$ for all $\gamma \in \Sigma$.

Set $\mathcal{A} = \oplus_{l(\lambda)=0} \mathcal{L}_\lambda$ and $\mathcal{B} = \oplus_{l(\lambda)\neq 0} \mathcal{L}_\lambda$. We have $\mathcal{L} = \mathcal{A} \oplus \mathcal{B}$ and $[\mathcal{A}, \mathcal{B}] \subset \mathcal{B}$. By Lemma 1.3, $\mathcal{B} + [\mathcal{B}, \mathcal{B}]$ is an ideal, and therefore $\mathcal{L} = \mathcal{B} + [\mathcal{B}, \mathcal{B}]$. Thus $\mathcal{A} \subset [\mathcal{B}, \mathcal{B}]$.

Since $l(\delta) = 0$, we have $L_\delta \in [\mathcal{B}, \mathcal{B}]$. Therefore there are $\gamma_1, \gamma_2 \in \Lambda$ with $l(\gamma_1) \neq 0$, $l(\gamma_2) \neq 0$ such that $[L_{\gamma_1}, L_{\gamma_2}]$ is a non-zero multiple of L_δ. Since $[L_\delta, L_{-\delta}] \neq 0$, $[[L_{\gamma_1}, L_{\gamma_2}], L_{-\delta}]$ is a non-zero multiple of L_0. Thus $[L_{\gamma_1}, [L_{\gamma_2}, L_{-\delta}]]$ or $[L_{\gamma_2}, [L_{\gamma_1}, L_{-\delta}]]$ is a non-zero multiple of L_0. By symmetry of the role of γ_1 and of γ_2, we can assume that $[L_{\gamma_1}, [L_{\gamma_2}, L_{-\delta}]] = cL_0$, where c is not zero. Thus γ_1 belongs to Σ.

Since $[L_{\gamma_1}, L_{\gamma_2}]$ is a non-zero multiple of L_δ, we get $< L^*_\delta | [L_{\gamma_1}, L_{\gamma_2}] > \neq 0$, and therefore $L_{\gamma_1}.L^*_\delta \neq 0$. Thus we have $\epsilon(\gamma_1) = -1$, which contradicts the hypothesis $\epsilon(\gamma) = 1$ for all $\gamma \in \Sigma$. $\qquad\square$

For any $\lambda \in \Lambda$, set $\Omega^*(\lambda) = \mathrm{Supp}\,\mathcal{L}.L^*_\lambda$. As a corollary of the Main Lemma, we get:

Lemma 14.9. *For any $\lambda \in \Lambda$, there is a finite set F such that $\Lambda = F + \Omega^*(\lambda)$.*

Proof. By the Main Lemma 14.8, we have $\Omega^*(0) \supset \{\mu \in \Lambda | l(\mu) \neq 0\}$. So we have $\Lambda = \Omega^*(0) \cup \alpha + \Omega^*(0)$, where α is any element with $l(\alpha) \neq 0$.

However by Lemma 1.4, we have $\Omega^*(\lambda) \equiv \Omega^*(0)$. Therefore we have $\Lambda = F + \Omega^*(\lambda)$, for some a finite subset F of Λ. $\qquad\square$

15   Local Lie algebras of rank 2

The aim of this section is Lemma 15.3, i.e. the fact that some local Lie algebras do not occur in a Lie algebra $\mathcal{L} \in \mathcal{G}$.

First, start with some definitions. Let L, S be Lie algebras. The Lie algebra S is called a *section* of L if there exists two Lie subalgebras G and R in L, such that R is an ideal of G and $G/R \simeq S$.

Let $\mathfrak{a}$ be another Lie algebra. The relative notions of a Lie $\mathfrak{a}$-algebra, an $\mathfrak{a}$-subalgebra, an $\mathfrak{a}$-ideal and an $\mathfrak{a}$-section are defined as follows. A *Lie $\mathfrak{a}$-algebra*

is a Lie algebra L on which $\mathfrak{a}$ acts by derivation. An $\mathfrak{a}$-*subalgebra* (respectively an $\mathfrak{a}$-ideal) of a Lie $\mathfrak{a}$-algebra L is a subalgebra (respectively an ideal) G of L which is stable by $\mathfrak{a}$. Let L, S be Lie $\mathfrak{a}$-algebras. The Lie algebra S is called an $\mathfrak{a}$-*section* of L if there exist an $\mathfrak{a}$-subalgebra G of L, and an $\mathfrak{a}$-ideal R of G such that $G/R \simeq S$ as a Lie $\mathfrak{a}$-algebra.

Following V.G. Kac [Ka1], a *local Lie algebra* is a quadruple $V = (\mathfrak{g}, V^+, V^-, \pi)$, where:

(i) $\mathfrak{g}$ is a Lie algebra and $V^\pm$ are $\mathfrak{g}$-modules

(ii) $\pi : V^+ \times V^- \to \mathfrak{g}$ is a $\mathfrak{g}$-equivariant bilinear map.

As for Lie algebras, there are obvious notions of *local subalgebras*, *local ideals* of V and *local sections* of V. By definition a *local Lie $\mathfrak{a}$-algebra* is a local Lie algebra $V = (\mathfrak{g}, V^+, V^-, \pi)$ such that $\mathfrak{g}$, V^+ and V^- are $\mathfrak{a}$-modules and all the products of the local structure are $\mathfrak{a}$-equivariant. As for Lie algebras, one defines the notion of a *local $\mathfrak{a}$-subalgebra*, of a *local $\mathfrak{a}$-ideal* and of a *local $\mathfrak{a}$-section* of V.

Let $\mathcal{L} = \oplus_{n \in \mathbb{Z}} \mathcal{L}_n$ be a weakly $\mathbb{Z}$-graded Lie algebra. The subspace $\mathcal{L}_{loc} = \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1$ is called its *local part*. It is clear that $\mathcal{L}_{loc}$ carries a structure of a local Lie algebra. Indeed $\mathfrak{g} = \mathcal{L}_0$ is a Lie algebra, $V^\pm = \mathcal{L}_{\pm 1}$ are $\mathcal{L}_0$-modules and the Lie bracket induces a bilinear map $\pi : \mathcal{L}_1 \times \mathcal{L}_{-1} \to \mathcal{L}_0$. By definition, the Lie algebra $\mathcal{L}$ is *associated to the local Lie algebra V* if $\mathcal{L}$ is generated by its local part and if $\mathcal{L}_{loc} \simeq V$.

Given a local Lie algebra V, there are two associated Lie algebras $\mathcal{L}^{max}(V)$ and $\mathcal{L}^{min}(V)$ which satisfies the following conditions. For any Lie algebra $\mathcal{L}$ associated to V, there are morphisms of Lie algebras $\mathcal{L}^{max} \to \mathcal{L}$ and $\mathcal{L} \to \mathcal{L}^{min}$. Here it should be understood that the local parts of these morphisms are just the identity.

Let $F(V^\pm)$ be the free Lie algebra generated by $V^\pm$. Indeed it is shown in [Ka1] that $F(V^+) \oplus \mathfrak{g} \oplus F(V^-)$ has a natural structure of a Lie algebra. It follows that $\mathcal{L}^{max}(V) = F(V^+) \oplus \mathfrak{g} \oplus F(V^-)$. The weak $\mathbb{Z}$-gradation of the Lie algebra $\mathcal{L} := \mathcal{L}^{max}(V)$ is described as follows: $\mathcal{L}_0 = \mathfrak{g}$, $\mathcal{L}_{\pm 1} = V^\pm$, so $F(V^+)$ is positively weakly graded and $F(V^-)$ is negatively weakly graded.

Let R be the maximal graded ideal of $\mathcal{L}^{max}(V)$ with the property that its local part $R_{-1} \oplus R_0 \oplus R_1$ is zero. Then we have $\mathcal{L}^{min}(V) = \mathcal{L}^{max}(V)/R$.

Let $\mathfrak{a}$ be an auxiliary Lie algebra. It should be noted that $\mathcal{L}^{max}(V)$ and $\mathcal{L}^{min}(V)$ are Lie $\mathfrak{a}$-algebras whenever V is a local Lie $\mathfrak{a}$-algebra.

Lemma 15.1. *Let $\mathcal{L}$ be a weakly $\mathbb{Z}$-graded algebra, and let V be an $\mathcal{L}_0$ local Lie algebra. If V is an $\mathcal{L}_0$-section of $\mathcal{L}_{loc}$, then $\mathcal{L}^{min}(V)$ is an $\mathcal{L}_0$-section of $\mathcal{L}$.*

Proof. By definition, there is a local $\mathcal{L}_0$-subalgebra U of $\mathcal{L}_{loc}$ and a local $\mathcal{L}_0$-ideal R of U such that $U/R \simeq V$ as a local Lie $\mathcal{L}_0$-algebra. Let $\mathcal{L}(U)$ be the subalgebra of $\mathcal{L}$ generated by U.

By definition, there is a surjective morphism $\mathcal{L}^{max}(U) \to \mathcal{L}(U)$ which is the identity on the local part. Since the positively graded part and the negatively graded part of $\mathcal{L}^{max}(V)$ are free Lie algebras, the local map $U \to V$ extends to a surjective map $\mathcal{L}^{max}(U) \to \mathcal{L}^{min}(V)$. It follows that $\mathcal{L}(U) = \mathcal{L}^{max}(U)/I$ and

$\mathcal{L}^{min}(V) = \mathcal{L}^{max}(U)/J$, where I and J are graded ideals. We have $I_{-1} \oplus I_o \oplus I_1 = 0$, and J is the maximal graded ideal such that $J_{-1} \oplus J_o \oplus J_1 \subset R$.

It follows that $I \subset J$. Thus $\mathcal{L}^{min}(V)$ is the quotient of the $\mathcal{L}_0$-subalgebra $\mathcal{L}(U)$ by its $\mathcal{L}_0$-ideal J/I. Hence $\mathcal{L}^{min}(V)$ is an $\mathcal{L}_0$-section of $\mathcal{L}$. $\qquad\square$

Recall that the tensor densities W-module Ω_s^δ have been defined in Section 12. Its elements are symbols $\sigma(f\partial^{-\delta})$, where $f \in z^{s-\delta}\mathbb{C}[z, z^{-1}]$. To simplify the notations, this symbol will be denoted by $f\partial^{-\delta}$. Let δ, η, s, t be four scalars. Define three maps:

$$\pi : \Omega_s^\delta \otimes \Omega_t^\eta \to \Omega_{s+t}^{\delta+\eta},\ f\partial^{-\delta} \otimes g\partial^{-\eta} \mapsto fg\partial^{-\delta-\eta},$$
$$\beta_1 : \Omega_s^\delta \otimes \Omega_t^\eta \to \Omega_{s+t}^{\delta+\eta+1},\ f\partial^{-\delta} \otimes g\partial^{-\eta} \mapsto fg'\partial^{-\delta-\eta-1},$$
$$\beta_2 : \Omega_s^\delta \otimes \Omega_t^\eta \to \Omega_{s+t}^{\delta+\eta+1},\ f\partial^{-\delta} \otimes g\partial^{-\eta} \mapsto f'g\partial^{-\delta-\eta-1}.$$

The map π is the product of symbols $P_{s,t}^{\delta,\eta}$. Denote by $\mathcal{K}$ its kernel. Since π is a morphism of W-modules, $\mathcal{K}$ is a W-submodule.

Lemma 15.2. *(i) We have $\beta_1(\omega) + \beta_2(\omega) = 0$ for any $\omega \in \mathcal{K}$.*

(ii) The restriction of β_1 to $\mathcal{K}$ is surjective, and it is a morphism of W-modules.

Proof. For $f\partial^{-\delta} \otimes g\partial^{-\eta} \in \Omega_s^\delta \otimes \Omega_t^\eta$ we have $(\beta_1+\beta_2)(f\partial^{-\delta} \otimes g\partial^{-\eta}) = (fg)'\partial^{-\delta-\eta-1}$, and therefore $\beta_1(\omega) + \beta_2(\omega) = 0$ for any $\omega \in \mathcal{K}$.

Note that $\eta\beta_2 - \delta\beta_1$ is the Poisson bracket $B_{s,t}^{\delta,\eta}$ of symbols, thus $\eta\beta_2 - \delta\beta_1$ is a morphism of W-modules. Since $\beta_1 + \beta_2 = 0$ on $\mathcal{K}$, the restriction of $(\delta + \eta)\beta_1$ to $\mathcal{K}$ is a morphism of W-module. Thus $\beta_1|_\mathcal{K}$ is a W-morphism when $\delta + \eta \neq 0$. By extension of polynomial identities, it is always true that $\beta_1|_\mathcal{K}$ is a morphism of W-modules.

Let $w \in (s + t - \delta - \eta) + \mathbb{Z}$. Choose $a \in (s - \delta) + \mathbb{Z}$, $b \in (t - \eta) + \mathbb{Z}$ with $a + b = w$. Set $\omega = z^a\partial^{-\delta} \otimes z^b\partial^{-\eta} - z^{a+1}\partial^{-\delta} \otimes z^{b-1}\partial^{-\eta}$. Then ω belongs to $\mathcal{K}$ and $\beta_1(\omega) = z^w\partial^{-\delta-\eta-1}$. Since the symbols $z^w\partial^{-\delta-\eta-1}$ form a basis $\Omega_{s+t}^{\delta+\eta+1}$, the restriction of β_1 to $\mathcal{K}$ is onto. $\qquad\square$

Let $V = (\mathfrak{g}, V^+, V^-)$ be a local Lie algebra with the following properties: $\mathfrak{g} = \oplus_x \mathfrak{g}_x$ is a $\mathbb{C}$-graded Lie algebra, $V^\pm = \oplus_x V_x^\pm$ are $\mathbb{C}$-graded $\mathfrak{g}$-modules and the bilinear map $\pi : V^+ \times V^- \to \mathfrak{g}$ is homogeneous of degree zero. Then the Lie algebras $\mathcal{L}^{max}(V)$ and $\mathcal{L}^{min}(V)$ are naturally weakly $\mathbb{Z} \times \mathbb{C}$-graded. Set $\mathcal{L} = \mathcal{L}^{min}(V)$ and denote by $\mathcal{L} = \oplus\mathcal{L}_{n,x}$ the corresponding decomposition. With the previous notations, let $\mathcal{F}_{loc}$ be the class of $\mathbb{C}$-graded local Lie algebras V such that

$$\dim \mathcal{L}_{n,x} \leq 1, \text{ for any } (n,x) \in \mathbb{Z} \times \mathbb{C}.$$

Since $W \simeq \Omega_0^{-1}$, the commutative product on $\mathcal{P}$ induces a W-equivariant bilinear map $\pi_s^{-\delta} : \Omega_s^{-\delta} \times \Omega_{-s}^{\delta-1} \to W$.

Lemma 15.3. *Assume that the local Lie algebra $(W, \Omega_s^{-\delta}, \Omega_{-s}^{\delta-1}, \pi_s^{-\delta})$ is in $\mathcal{F}_{loc}$. Then $\delta = -1$ or $\delta = 2$.*

48

Proof. For clarity, the proof is divided into four steps. It is assumed, once for all, that $\delta \neq -1$ and $\delta \neq 2$.

Step 1: Let V be any local Lie algebra. Set $\mathcal{L} = \mathcal{L}^{max}(V)$ and let R be the kernel of the morphism $\mathcal{L} \to \mathcal{L}^{min}(V)$.

In order to compute by induction, for $n \geq 1$, the homogeneous components $\mathcal{L}_n/R_n$ of $\mathcal{L}^{min}(V)$, it should be noted that:

(i) $\mathcal{L}_{\geq 1} := \oplus_{n \geq 1} \mathcal{L}_n$ is the Lie algebra freely generated by $V^+ = \mathcal{L}_1$, and therefore $\mathcal{L}_{n+1} = [\mathcal{L}_1, \mathcal{L}_n]$ for all $n \geq 1$,

(ii) $R_1 = 0$ and $R_{n+1} = \{x \in \mathcal{L}_{n+1} | [\mathcal{L}_{-1}, x] \subset R_n\}$ for any $n \geq 1$.

More precisely, the following procedure will be used. Assume by induction that:

(i) the $\mathcal{L}_0$-modules $\mathcal{L}_i/R_i$,

(ii) the brackets $[\ ,\ ] : \mathcal{L}_1 \times \mathcal{L}_{i-1} \to \mathcal{L}_i$,

(iii) the brackets $[\ ,\ ] : \mathcal{L}_{-1} \times \mathcal{L}_i \to \mathcal{L}_{i-1}$

have been determined for all $1 \leq i \leq n$.

Thus define the bilinear map: $B_n : \mathcal{L}_{-1} \times \mathcal{L}_1 \otimes \mathcal{L}_n/R_n \to \mathcal{L}_n/R_n$ by the formula $B_n(x, y \otimes z) = [[x,y],z] + [y,[x,z]]$, for any $x \in \mathcal{L}_{-1}, y \in \mathcal{L}_1$ and $z \in \mathcal{L}_n/R_n$. Let $K \subset \mathcal{L}_1 \otimes \mathcal{L}_n/R_n$ be the right kernel of B_n. It follows from Jacobi identity that the following diagram is commutative:

By definition of $\mathcal{L}^{min}(V)$, the right kernel of the horizontal bilinear map: $[,] : \mathcal{L}_{-1} \times \mathcal{L}_{n+1}/R_{n+1} \to \mathcal{L}_n/R_n$ is zero. Moreover the natural map $[,] : \mathcal{L}_1 \otimes \mathcal{L}_n/R_n \to \mathcal{L}_{n+1}/R_{n+1}$ is onto. Therefore, we have:

$$\mathcal{L}_{n+1}/R_{n+1} = (\mathcal{L}_1 \otimes \mathcal{L}_n/R_n)/K$$

This isomorphism determines the structure of $\mathcal{L}_0$-module of $\mathcal{L}_{n+1}/R_{n+1}$ as well as the bracket $[,] : \mathcal{L}_1 \times \mathcal{L}_n \to \mathcal{L}_{n+1}$. The bracket $[,] : \mathcal{L}_{-1} \times \mathcal{L}_{n+1} \to \mathcal{L}_n$ comes from B_n.

Step 2: The local Lie algebras $V := (W, \Omega_s^{-\delta}, \Omega_{-s}^{\delta-1}, -\pi_s^{-\delta})$ and $(W, \Omega_s^{-\delta}, \Omega_{-s}^{\delta-1}, \pi_s^{-\delta})$ are obviously isomorphic. In order to minimize the use of minus sign, it will be more convenient to work with V.
Set $\mathcal{L} = \mathcal{L}^{max}(V)$ and set $A = \mathbb{C}[z, z^{-1}]$, $A_k = z^{k(s+\delta)} A$ for any $k \in \mathbb{Z}$. We will use the map B_1 of step 1 to compute $\mathcal{L}_2/R_2$.

The elements of $\mathcal{L}_1$ (respectively of $\mathcal{L}_{-1}$) are symbols $f\partial^\delta$ (respectively $g\partial^\gamma$), where $f \in A_1$ (respectively where $g \in A_{-1}$) and where $\gamma = 1 - \delta$. In the local Lie algebra V, we have:

$$[g\partial^\gamma, f\partial^\delta] = \pi(f\partial^\delta, g\partial^\gamma) = fg\partial \in W.$$

Let $f\partial^\delta$, $g\partial^\delta \in \mathcal{L}_1$ and let $h\partial^\gamma \in \mathcal{L}_{-1}$. It follows that:

$$
\begin{aligned}
B_1(h\partial^\gamma, f\partial^\delta \otimes g\partial^\delta) &= [hf\partial, g\partial^\delta] + [f\partial^\delta, hg\partial] \\
&= (hfg' - \delta g(hf)' + \delta f(hg)' - f'hg)\partial^\delta \\
&= (1 + \delta)h(fg' - f'g)\partial^\delta
\end{aligned}
$$

Define $\beta : \mathcal{L}_1 \otimes \mathcal{L}_1 \to \Omega_{2s}^{1-2\delta}$ by the formula $\beta(f\partial^\delta \otimes g\partial^\delta) = (1 + \delta)(fg' - f'g)\partial^{2\delta-1}$. In terms of Poisson brackets, we have $\beta(b \otimes c) = (1 + \delta)/\delta\{b, c\}$ for any $b, c \in \Omega_s^\delta$, and therefore β is a morphism of W-modules (for $\delta = 0$, this follows by extension of a polynomial identity). Moreover it is easy to show that β is surjective.

Since $\delta \neq -1$, it is clear from the previous formula that the right kernel of B_1 is precisely the kernel of β. So we get $\mathcal{L}_2/R_2 \simeq \Omega_{2s}^{1-2\delta}$.

One can choose such an isomorphism in a way that:

$[f\partial^\delta, g\partial^\delta] = (1 + \delta)(fg' - f'g)\partial^{2\delta-1}$,
$[h\partial^\gamma, k\partial^{2\delta-1}] = hk\partial^\delta$,

for any $f\partial^\delta$, $g\partial^\delta \in \mathcal{L}_1$, $h\partial^\gamma \in \mathcal{L}_{-1}$ and $k\partial^{1-2\delta} \in \mathcal{L}_2/R_2$.

Step 3: We will use the map B_2 of step 1 to compute $\mathcal{L}_3/R_3$.

Let $f\partial^\delta \in \mathcal{L}_1$, $k\partial^{2\delta-1} \in \mathcal{L}_2/R_2$ and let $h\partial^\gamma \in \mathcal{L}_{-1}$. We get

$$
\begin{aligned}
B_2(h\partial^\gamma, f\partial^\delta \otimes k\partial^{2\delta-1}) &= [hf\partial, k\partial^{2\delta-1}] + [f\partial^\delta, hk\partial^\delta] \\
&= (hfk' - (2\delta - 1)(hf)'k + (1 + \delta)(f(hk)' - f'hk))\partial^{2\delta-1} \\
&= ((2 - \delta)h'fk - 3\delta hf'k + (2 + \delta)hfk')\partial^{2\delta-1} \\
&= ((2 - \delta)fk)h'\partial^{2\delta-1} + (-3\delta f'k + (2 + \delta)fk')h\partial^{2\delta-1}.
\end{aligned}
$$

Since $\delta \neq 2$, it is clear that the right kernel of B_2 is

$$\mathrm{Ker}\,\pi \cap \mathrm{Ker}\,((2 + \delta)\beta_1 - 3\delta\beta_2),$$

where $\pi : \Omega_s^{-\delta} \otimes \Omega_{2s}^{1-2\delta} \to \Omega_{3s}^{1-3\delta}$ and $\beta_1, \beta_2 : \Omega_s^{-\delta} \otimes \Omega_{2s}^{1-2\delta} \to \Omega_{3s}^{2-3\delta}$ are defined in Lemma 15.2. However we have $((2 + \delta)\beta_1 - 3\delta\beta_2)(\omega) = (4\delta + 2)\beta_1(\omega)$ for all $\omega \in \mathrm{Ker}\,\pi$.

Assume now that $\delta \neq -1/2$, i.e. $4\delta + 2 \neq 0$. It follows that $\mathcal{L}_3/R_3 \simeq [\mathcal{L}_1 \otimes \mathcal{L}_2/R_2]/\mathrm{Ker}\,\pi \cap \mathrm{Ker}\,\beta_1$ and there is an exact sequence:

$0 \to \mathrm{Ker}\,\pi/\mathrm{Ker}\,\beta_1 \cap \mathrm{Ker}\,\pi \to \mathcal{L}_3/R_3 \to [\mathcal{L}_1 \otimes \mathcal{L}_2/R_2]/\mathrm{Ker}\,\pi \to 0$. Using Lemma 15.2, we get an exact sequence:

$$0 \to \Omega_{3s}^{2-3\delta} \to \mathcal{L}_3/R_3 \to \Omega_{3s}^{1-3\delta} \to 0.$$

Thus, the homogeneous components of $\mathcal{L}_3/R_3$ have dimension two. It follows that the local Lie algebra V is not in $\mathcal{F}_{loc}$ if $\delta \neq -1/2$.

Step 4: Assume now that $\delta = -1/2$. The opposite local Lie algebra is $V' = (W, \Omega_{-s}^{-3/2}, \Omega_s^{1/2}, \pi_{-s}^{-3/2})$. It follows from the previous step that the homogeneous components of $\mathcal{L}_{-3}/R_{-3}$ have dimension two. Thus the local Lie algebra V is not in $\mathcal{F}_{loc}$. $\qquad\square$

16 The degree function δ

Let $\mathcal{L} \in \mathcal{G}$ be non-integrable.

By Lemma 14.8, we have $\Sigma = \{\alpha \,|\, l(\alpha) \neq 0\}$. Therefore Σ contains primitive elements of Λ. So fix, once for all, a primitive $\alpha \in \Lambda$ which is in Σ. Set $\mathcal{L}(\alpha) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{n\alpha}$ and, for any $\beta \in \Lambda/\mathbb{Z}\alpha$, set $\mathcal{M}(\beta) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{\beta + n\alpha}$.

By Lemma 11.4, the Lie algebra $\mathcal{L}(\alpha)$ is isomorphic to W. Since $\mathcal{M}(\beta)$ belongs to the class $\mathcal{S}(W)$, the map $\beta \in \Lambda/\mathbb{Z}\alpha \mapsto \deg \mathcal{M}(\beta)$ is a multivalued function. In this section, Lemma 16.4 is used to define an ordinary function $\delta :$ $\Lambda/\mathbb{Z}\alpha \to \mathbb{C}$, which has the property that $\delta(\beta) \in \deg \mathcal{M}(\beta)$ for all β. Additional properties of δ will be proved in this section and the next one.

A weakly $\mathbb{Z}$-graded Lie algebra $L = \oplus_{n \in \mathbb{Z}} L_n$ is called *minimal* iff it satisfies the following conditions:

(i) L is generated by its local part $L_{loc} := L_{-1} \oplus L_0 \oplus L_1$

(ii) Any graded ideal I such that $I \cap L_{loc} = 0$ is trivial.

Equivalently, $L = \mathcal{L}^{min}(V)$ where V is the local part of L.

For any integer a, the notation $\mathbb{Z}_{>a}$ has been defined in Section 1.1. It will be convenient to extend this notation for $a = -\infty$. In this case, set $\mathbb{Z}_{>-\infty} = \mathbb{Z}$.

Lemma 16.1. *Let $a \in \mathbb{Z} \cup \{-\infty\}$ and let $L = \oplus_{n \in \mathbb{Z}_{>a}} L_n$ be a weakly graded Lie algebra. Assume that L_n is a simple L_0-module, $[L_{-1}, L_{n+1}] \neq 0$ and $[L_1, L_n] \neq 0$, for any $n > a$. Then L is minimal.*

Proof. Let L' be the subalgebra generated by L_{loc}.

Let R be any non-zero graded L'-submodule of L and let $n > a$. By simplicity of the L_0-module L_n, we have $[L_1, L_n] = L_{n+1}$. Similarly, we have $[L_{-1}, L_n] = L_{n-1}$ (for $n = a + 1$, this follows from $L_a = 0$). So, for $n \in \mathrm{Supp}\, R$, we have $R_n = L_n$ and $n+1 \in \mathrm{Supp}\, R$. Moreover if $n-1 > a$, we also have $n-1 \in \mathrm{Supp}\, R$. Thus $\mathrm{Supp}\, R = \mathbb{Z}_{>a}$ and $R = L$.

It follows that $L = L'$, i.e. L is generated by its local part, and that L is a simple graded Lie algebra. Therefore L is minimal. $\qquad\square$

Recall that $E_{-\rho} \in \mathcal{P}$ represents the symbol of 1. The center of the Lie algebras $\mathcal{P}^+$ and $\mathcal{P}$ is $\mathbb{C}E_{-\rho}$. Thus set $\widetilde{\mathcal{P}}^+ = \mathcal{P}^+/\mathbb{C}.E_{-\rho}$ and $\widetilde{\mathcal{P}} = [\mathcal{P}, \mathcal{P}]/\mathbb{C}E_{-\rho}$. Since $\mathcal{P} = [\mathcal{P}, \mathcal{P}] \oplus \mathbb{C}E_{-2\rho}$, the Lie algebra $\widetilde{\mathcal{P}}$ has basis (E_λ) when λ runs over $\mathbb{C}^2 \setminus \{-\rho, -2\rho\}$, and the bracket is defined as before by:

$[E_\lambda, E_\mu] = 0$ if $\lambda + \mu = -\rho$ or -2ρ

$[E_\lambda, E_\mu] = <\lambda + \rho \,|\, \mu + \rho> E_{\lambda+\mu}$ otherwise,

for any $\lambda, \mu \in \mathbb{C}^2 \setminus \{-\rho, -2\rho\}$.

Set $Par^+ = \mathbb{Z}_{\geq 0} \times \mathbb{C}/\mathbb{Z}$, $Par = \mathbb{C} \times \mathbb{C}/\mathbb{Z}$. As a W-module, there are decompositions:

$$\mathcal{P}^+ = \oplus_{(\delta, u) \in Par^+} \Omega_u^\delta \text{ and } \mathcal{P} = \oplus_{(\delta, u) \in Par} \Omega_u^\delta$$

Accordingly, there are decompositions of $\widetilde{\mathcal{P}}^+$ and $\widetilde{\mathcal{P}}$:

$$\widetilde{\mathcal{P}}^+ = \oplus_{(\delta, u) \in Par^+} \widetilde{\Omega}_u^\delta \text{ and } \widetilde{\mathcal{P}} = \oplus_{(\delta, u) \in Par} \widetilde{\Omega}_u^\delta,$$

where: $\widetilde{\Omega}_0^0 = \Omega_0^0/\mathbb{C}$, $\widetilde{\Omega}_0^1 = d\Omega_0^0 \simeq \Omega_0^0/\mathbb{C}$, and $\widetilde{\Omega}_u^\delta = \Omega_u^\delta$ for $(\delta, u) \neq (0,0)$ or $(1,0)$.

In $\mathcal{P}$, we have:

$$\{\Omega_u^\delta, \Omega_{u'}^{\delta'}\} \subset \Omega_{u+u'}^{\delta+\delta'+1}.$$

Similarly, in $\widetilde{\mathcal{P}}$ we have

$$\{\widetilde{\Omega}_u^\delta, \widetilde{\Omega}_{u'}^{\delta'}\} \subset \widetilde{\Omega}_{u+u'}^{\delta+\delta'+1}.$$

Let $(\delta, u) \in Par$. Set $\widetilde{\mathcal{P}}_{\delta,u}^n = \tilde{\Omega}_{nu}^{n(\delta+1)-1}$ and $\widetilde{\mathcal{P}}_{\delta,u} = \oplus_{n \in \mathbb{Z}} \widetilde{\mathcal{P}}_{\delta,u}^n$. It follows that $\widetilde{\mathcal{P}}_{\delta,u}$ is a Lie subalgebra of $\widetilde{\mathcal{P}}$. Moreover the decomposition $\widetilde{\mathcal{P}}_{\delta,u} = \oplus_{n \in \mathbb{Z}} \widetilde{\mathcal{P}}_{\delta,u}^n$ is a weak $\mathbb{Z}$-gradation of the Lie algebra.

Similarly, for $u \in \mathbb{C}/\mathbb{Z}$, set $\widetilde{\mathcal{P}}_n^+(u) = \widetilde{\Omega}_{nu}^{-n-1}$ and $\widetilde{\mathcal{P}}^+(u) = \oplus_{n \geq -1} \widetilde{\mathcal{P}}_n^+(u)$. It is clear that $\widetilde{\mathcal{P}}^+(u)$ is a weakly $\mathbb{Z}$-graded subalgebra of $\widetilde{\mathcal{P}}^+$.

Lemma 16.2. *(i) Let $(\delta, u) \in Par$ with $\delta \neq 0$ or -2. Then the weakly $\mathbb{Z}$-graded Lie algebra $\widetilde{\mathcal{P}}_{\delta,u}$ is minimal,*

(ii) Let $u \in \mathbb{C}/\mathbb{Z}$. Then the weakly $\mathbb{Z}$-graded Lie algebra $\widetilde{\mathcal{P}}^+(u)$ is minimal.

Proof. It should be noted that

(i) $\widetilde{\Omega}_u^\delta$ is a simple W-module for all $(\delta, u) \in Par$,

(ii) $\{\widetilde{\Omega}_u^\delta, \widetilde{\Omega}_{u'}^{\delta'}\} \neq 0$, except when $\delta = \delta' = 0$.

It follows that the Lie algebra $\widetilde{\mathcal{P}}_{\delta,u} = \oplus_{n \in \mathbb{Z}} \widetilde{\mathcal{P}}_{\delta,u}^n$ satisfies the hypothesis of the previous lemma for $a = -\infty$. Similarly, the Lie algebra $\widetilde{\mathcal{P}}^+(u) = \oplus_{n \geq -1} \widetilde{\mathcal{P}}_n^+(u)$ satisfies the hypothesis of the previous lemma for $a = -2$. Thus these Lie algebras are minimal. $\square$

For $\beta \in \Lambda/\mathbb{Z}\alpha$, denote by $V(\beta)$ the local Lie algebra $\mathcal{M}(-\beta) \oplus \mathcal{L}(\alpha) \oplus \mathcal{M}(\beta)$.

Lemma 16.3. *There exists $(\delta, s) \in Par$ such that the local Lie algebra $V(\beta)$ admits $\widetilde{\mathcal{P}}_{\delta,s}^{loc}$ as a W-section.*

Proof. Let $\mu : \mathcal{M}(\beta) \times \mathcal{M}(-\beta) \to \mathcal{L}(\alpha) \simeq W$ be the W-equivariant bilinear map induced by the bracket.

Choose any $\gamma \in \beta$ with $l(\gamma) \neq 0$. By Lemma 11.4, γ belongs to Σ, hence $\mu(L_\gamma, L_{-\gamma}) \neq 0$. It follows from Lemma 13.3 that μ is non-degenerate. However all W-equivariant non-degenerate bilinear maps $\mu : M \times N \to W$ are classified by Lemma 13.5. Indeed μ is either conjugated to the product of symbols $\pi_s^\delta : \Omega_s^\delta \times \Omega_{-s}^{-1-\delta} \to \Omega_0^{-1} \simeq W$, for $\delta \neq 1$ or -2, to the Poisson bracket of symbols $\beta_s^\delta : \Omega_s^\delta \times \Omega_{-s}^{-2-\delta} \to \Omega_0^{-1} \simeq W$, for $(\delta, s) \neq (0,0)$ or $(-2,0)$ or μ is in the complementary family.

By Lemma 15.1, $\mathcal{L}^{min}(V(\beta))$ is a section of $\mathcal{L}$. Therefore $V(\beta)$ should belong to the class $\mathcal{F}_{loc}$. It follows from Lemma 15.3 that μ cannot be proportional to the map π_s^δ, for $\delta \neq 1$ or -2.

When μ is proportional to β_s^δ, then $V(\beta)$ is isomorphic to $\widetilde{\mathcal{P}}_{\delta,s}^{loc}$ if $(\delta, s) \neq (0,0), (-2,0), (1,0), (-3,0)$. Assume now that μ is proportional to β_0^1 (respectively β_0^{-3}). Note that $\tilde{\Omega}_0^1 = d\Omega_0^0$ is a codimension one W-submodule of Ω_0^1,

therefore $\widetilde{\mathcal{P}}_{1,0}^{loc}$ (respectively $\widetilde{\mathcal{P}}_{-3,0}^{loc}$) is a codimension one ideal of $V(\beta)$. Thus $V(\beta)$ admits a W-section isomorphic to some $\widetilde{\mathcal{P}}_{\delta,s}^{loc}$, whenever μ is proportional to some β_s^δ, with $(\delta, s) \neq (0,0)$ or $(-2,0)$.

When μ is in the left complementary family, the W-module $\mathcal{M}(\beta)$ belongs to the AB-family and then there is an exact sequence
$$0 \to \widetilde{\Omega}_0^0 \to \mathcal{M}(\beta) \to \mathbb{C} \to 0, \text{ or } 0 \to \mathbb{C} \to \mathcal{M}(\beta) \to \widetilde{\Omega}_0^0 \to 0.$$
In the first case, $V(\beta)$ has a codimension one ideal isomorphic to $\widetilde{\mathcal{P}}_{0,0}^{loc}$. In the second case, the trivial submodule of $\mathcal{M}(\beta)$ is the center $\mathfrak{z}$ of the local Lie algebra $V(\beta)$, and $V(\beta)/\mathfrak{z}$ is isomorphic to $\widetilde{\mathcal{P}}_{0,0}^{loc}$. In both cases, $\widetilde{\mathcal{P}}_{0,0}^{loc}$ is a W-section of $V(\beta)$.

Similarly when μ is in the right complementary family, $\widetilde{\mathcal{P}}_{-2,0}^{loc}$ is a W-section of $V(\beta)$.

Thus the lemma is proved in all cases. $\qquad\square$

Lemma 16.4. *Let $\beta \in \Lambda/\mathbb{Z}\alpha$. There exists a unique scalar $\delta(\beta)$ such that*

(i) $\delta(\beta) \in \deg \mathcal{M}(\beta)$, *and*

(ii) $-\delta(\beta) - 2 \in \deg \mathcal{M}(-\beta)$.

Proof. Let $\beta \in \Lambda/\mathbb{Z}\alpha$. Note that for $\beta = 0$, then $\delta(0) = -1$ is the unique degree of $\mathcal{M}(0) \simeq W$ and it satisfies (ii). From now on, it can be assumed that $\beta \neq 0$.

First prove the existence of $\delta(\beta)$.

By Lemma 16.3, there exists $(\delta, u) \in Par$ such that the local Lie algebra $V(\beta)$ admits $\widetilde{\mathcal{P}}_{\delta,u}^{loc}$ as W-section.

Hence the W-module $\widetilde{\Omega}_u^\delta$ is a subquotient of $\mathcal{M}(\beta)$ and the W-module $\widetilde{\Omega}_{-u}^{-\delta-2}$ is a subquotient of $\mathcal{M}(-\beta)$. Thus $\delta \in \deg \mathcal{M}(\beta)$ and $-\delta - 2 \in \deg \mathcal{M}(-\beta)$. The scalar $\delta(\beta) = \delta$ satisfies (i) and (ii), and the existence is proved.

Next prove the unicity. If $\deg \mathcal{M}(\beta)$ is single valued, then $\delta(\beta)$ is uniquely determined. Assume otherwise. Then $\deg \mathcal{M}(\beta) = \{0, 1\}$. It follows from the previous point that -2 belongs to $\deg \mathcal{M}(\beta) + \deg \mathcal{M}(-\beta)$. Therefore -2 or -3 is a degree of $\mathcal{M}(-\beta)$. So $\deg \mathcal{M}(-\beta)$ is single valued, and $-2 - \delta(\beta)$ is uniquely determined. In both case, $\delta(\beta)$ is uniquely determined. $\qquad\square$

It follows from the previous lemma that there is a well determined function $\delta : \Lambda/\mathbb{Z}\alpha \to \mathbb{C}, \beta \mapsto \delta(\beta)$ with the property that $\delta(\beta) \in \deg \mathcal{M}(\beta)$ and $\delta(\beta) + \delta(-\beta) = -2$. This function will be called the *degree function*.

Lemma 16.5. *Let $\beta \in \Lambda/\mathbb{Z}\alpha$. Then we have:*
$$\delta(n\beta) = n(\delta(\beta) + 1) - 1, \; \forall n \in \mathbb{Z}$$

Proof. Note that $\delta(0) = -1$, so we may assume that $\beta \neq 0$. Set $\mathcal{M} = \oplus_{n \in \mathbb{Z}} \mathcal{M}_n$, where $\mathcal{M}_n = \mathcal{M}(n\beta)$ for all integer n. Then $\mathcal{M} \subset \mathcal{L}$ is a weakly $\mathbb{Z}$-graded Lie algebra.

First assume that $\delta(\beta) \neq 0$ or -2. Set $d(n) = n(\delta(\beta) + 1) - 1$. By Lemma 16.3, the local Lie algebra $V(\beta) = \mathcal{M}(-\beta) \oplus \mathcal{L}(\alpha) \oplus \mathcal{M}(\beta)$ admits $\widetilde{\mathcal{P}}_{\delta(\beta),u}^{loc}$ as a W-section, for some $u \in \mathbb{C}/\mathbb{Z}$. By Lemma 16.2, $\widetilde{\mathcal{P}}_{\delta(\beta),u}$ is minimal. Thus

by Lemma 15.1, $\widetilde{\mathcal{P}}_{\delta(\beta),u}$ is a W-section of $\mathcal{M}$. Hence the W-module $\widetilde{\Omega}_{nu}^{d(n)}$ is a subquotient of $\mathcal{M}(n\beta)$. So we have $d(\pm n) \in \deg \mathcal{M}(\pm n\beta)$.

Moreover, it is obvious that $d(n) + d(-n) = -2$. Therefore it follows from Lemma 16.4 that $d(n) = \delta(n\beta)$, and the lemma is proved in this case.

Next assume that $\delta(\beta) = -2$. By Lemma 16.3, the local Lie algebra $V(\beta)$ admits $\widetilde{\mathcal{P}}_{-2,u}^{loc}$ as a W-section, for some $u \in \mathbb{C}/\mathbb{Z}$. However $\widetilde{\mathcal{P}}_{-2,u}^{loc}$ is the local part of weakly $\mathbb{Z}$-graded Lie algebra $\widetilde{\mathcal{P}}^{+}(u)$, which is minimal by Lemma 16.2. Hence $\widetilde{\mathcal{P}}^{+}(u)$ is a W-section of $\mathcal{M}$. Hence the W-module $\widetilde{\Omega}_{nu}^{-n-1}$ is a subquotient of $\mathcal{M}(n\beta)$ for all $n \geq -1$. It follows that $\deg \mathcal{M}(n\beta)$ is single valued for all $n \geq 0$, and that $\deg \mathcal{M}(n\beta) = -n-1$. Thus we get $\delta(n\beta) = -n-1$ for all $n \geq 0$ Since $\delta(n\beta) + \delta(-n\beta) = -2$, it follows that $\delta(n\beta) = -n-1$ for all $n \in \mathbb{Z}$, and the lemma is proved in this case.

The last case $\delta(\beta) = 0$ is identical to the previous one, because $\delta(-\beta) = -2$. $\qquad\square$

17 The degree function δ is affine

As before, fix once for all, a primitive element $\alpha \in \Lambda$ such that $l(\alpha) \neq 0$. It follows from previous considerations that the Lie subalgebra $\mathcal{L}(\alpha) = \oplus_{n\in\mathbb{Z}} \mathcal{L}_{n\alpha}$ is isomorphic to W. For simplicity, any $\mathbb{Z}\alpha$-coset $\beta \in \Lambda/\mathbb{Z}\alpha$ will be called a *coset*. For a coset β, set $\mathcal{M}(\beta) = \oplus_{\gamma\in\beta} \mathcal{L}_{\gamma}$. We have

$$[\mathcal{M}(\beta_1), \mathcal{M}(\beta_2)] \subset \mathcal{M}(\beta_1 + \beta_2)$$

for any cosets β_1, β_2.

Lemma 17.1. *Let β, γ be cosets. Assume that $\delta(\beta) \neq 0$. Then we have*
$$[\mathcal{M}(\beta), \mathcal{M}(\gamma)] \neq 0.$$

Proof. Assume that $\delta(\beta) \neq 0$. Set $\mathcal{M} = \oplus_{n\in\mathbb{Z}} \mathcal{M}_n$, where $\mathcal{M}_n = \mathcal{M}(n\beta)$ for all integer n. Then $\mathcal{M}$ is a weakly $\mathbb{Z}$-graded Lie algebra.

Step 1: We claim that:
$$[\mathcal{M}(\beta), \mathcal{M}(\beta)] \supset [L_0, \mathcal{M}(2\beta)].$$

By Lemma 16.3, the local Lie algebra $V(\beta) := \mathcal{M}(-\beta)\oplus\mathcal{L}(\alpha)\oplus\mathcal{M}(\beta)$ admits $\widetilde{\mathcal{P}}_{\delta(\beta),u}^{loc}$ as a W-section, for some $u \in \mathbb{C}/\mathbb{Z}$.

If $\delta(\beta) \neq -2$, the weakly $\mathbb{Z}$-graded Lie algebra $\widetilde{\mathcal{P}}_{\delta(\beta),u}$ is minimal. Thus by Lemma 15.1, $\widetilde{\mathcal{P}}_{\delta(\beta),u}$ is a W-section of $\mathcal{M}$. Similarly the weakly $\mathbb{Z}$-graded Lie algebra $\widetilde{\mathcal{P}}^{+}(u)$ is the minimal Lie algebra associated to the local Lie algebra $\widetilde{\mathcal{P}}_{-2,u}^{loc}$. Thus $\widetilde{\mathcal{P}}^{+}(u)$ is a W-section of $\mathcal{M}$ if $\delta(\beta) = -2$

In both case, we have $[\widetilde{\Omega}_u^{\delta(\beta)}, \widetilde{\Omega}_u^{\delta(\beta)}] = \widetilde{\Omega}_{2u}^{2\delta(\beta)+1}$. Since $\widetilde{\Omega}_{2u}^{2\delta(\beta)+1}$ is a W-subquotient of $\mathcal{M}_2$, the $\mathbb{C}$-graded vector spaces $[L_0, \widetilde{\Omega}_{2u}^{2\delta(\beta)+1}]$ and $[L_0, \mathcal{M}(2\beta)]$ are isomorphic, the claim follows.

Step 2: Let $M \in \mathcal{S}(W)$ and let $m \in M$ be a non-zero vector such that $L_0.m = xm$ for some $x \neq 0$. Then we have
$$L_1.m \neq 0 \text{ or } L_2.m \neq 0.$$

This claim follows easily from Kaplansky-Santharoubane Theorem.

Step 3: Fix $\beta_0 \in \beta$ and $\gamma_0 \in \gamma$ with $l(\beta_0) \neq 0$ and $l(\gamma_0) \neq 0$. We claim that
$$[L_{\beta_0}, L_{\gamma_0}] \neq 0 \text{ or } [L_{2\beta_0}, L_{\gamma_0}] \neq 0.$$

By Lemma 14.8, β_0 lies in Σ, and by Lemma 11.4 the Lie subalgebra $\mathcal{L}(\beta_0) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_{n\beta_0}$ is isomorphic to W. Thus the $\mathcal{L}(\beta_0)$-module
$$\mathcal{M}(\beta_0, \gamma_0) := \oplus_{n \in \mathbb{Z}} \mathcal{L}_{\gamma_0 + n\beta_0},$$
belongs to $\mathcal{S}(W)$. Since $l(\gamma_0) \neq 0$, it follows from Step 2 that $[L_{\beta_0}, L_{\gamma_0}] \neq 0$ or $[L_{2\beta_0}, L_{\gamma_0}] \neq 0$.

Final step: Let β_0 and γ_0 as before. By definition, L_{β_0} belongs to $\mathcal{M}(\beta)$. By Step 1, $[\mathcal{M}(\beta), \mathcal{M}(\beta)]$ contains $L_{2\beta_0}$. Thus by the previous claim, we have $[\mathcal{M}(\beta), L_{\gamma_0}] \neq 0$ or $[[\mathcal{M}(\beta), \mathcal{M}(\beta)], L_{\gamma_0}] \neq 0$. In both cases, we have $[\mathcal{M}(\beta), L_{\gamma_0}] \neq 0$ and therefore $[\mathcal{M}(\beta), \mathcal{M}(\gamma)] \neq 0$. $\square$

Lemma 17.2. *Let β, γ be cosets. We have:*
$$\delta(\beta + \gamma) = \delta(\beta) + \delta(\gamma) + 1.$$

Proof. Step 1: Let η be any coset. By Lemma 16.5, we have
$$\delta(n\eta) = n(\delta(\eta) + 1) - 1,$$
so we have
$$\delta(n\eta) \equiv -1 \text{ for all } n \text{ or } |\delta(n\eta)| \to \infty \text{ when } n \to \infty.$$
So there exists $k \neq 0$ such that the set $\{\delta(k\beta), \delta(-k\beta), \delta(k\gamma), \delta(k(\beta + \gamma))\}$ contains neither 0 nor 1.

Step 2: Let k as before and let
$\mu^+ : \mathcal{M}(k\beta) \times \mathcal{M}(k\gamma) \to \mathcal{M}(k(\beta + \gamma))$, and
$\mu^- : \mathcal{M}(-k\beta) \times \mathcal{M}(k(\beta + \gamma)) \to \mathcal{M}(k\gamma)$
be the W-equivariant bilinear maps induced by the Lie bracket. By Lemma 17.1, $\mu^{\pm}$ are non-zero. We have

$$\deg \mu^+ + \deg \mu^- = [\delta(k(\beta + \gamma)) - \delta(k\beta) - \delta(k\gamma)] + [\delta(k\gamma) - \delta(-k\beta) - \delta(k(\beta + \gamma))]$$
$$= -\delta(k\beta) - \delta(-k\beta).$$

So it follows from Lemma 16.4 that
$$\deg \mu^+ + \deg \mu^- = 2.$$
Observe that, in the list of Lemma 13.1, the bilinear maps of degree $-2, -1$ or 2 involves at least one module of degree 0 or 1. Thus it follows from the choice of k that the degrees of $\mu^{\pm}$ are $0, 1$ or 3. So the only solution of the previous equation is:
$$\deg \mu^+ = \deg \mu^- = 1,$$
and therefore $\delta(k(\beta + \gamma)) = \delta(k\beta) + \delta(k\gamma) + 1$. It follows from Lemma 16.5 that:
$$k(\delta(\beta + \gamma) + 1) - 1 = [k(\delta(\beta) + 1) - 1] + [k(\delta(\gamma) + 1) - 1] + 1,$$
from which the relation $\delta(\beta + \gamma) = \delta(\beta) + \delta(\gamma) + 1$ follows. $\square$

Lemma 17.3. *Let M, N in $\mathcal{S}(W)$.*

(i) *If there is a non-zero W-morphism $\mu : M \to N$, then $\deg M = \deg N$.*

(ii) *If there is a non-zero W-invariant bilinear map $\nu : M \times N \to \mathbb{C}$, then $\deg M = 1 - \deg N$.*

Proof. If μ is an isomorphism, the assertion is clear. Otherwise both M and N are reducible, and $\deg M = \deg N = \{0, 1\}$. Thus Assertion (i) is proved. The bilinear map ν gives rise to a non-zero morphism $\nu^* : M \to N'$, thus we get $\deg M = \deg N' = 1 - \deg N$. $\square$

Recall that $\mathcal{L}(\alpha)$ is identified with W.

Lemma 17.4. *For any coset β, the W-module $\mathcal{M}(\beta)$ is irreducible.*

Proof. Step 1: Let $x \in \mathbb{C}$, $x \neq -1$, and set $\Lambda_x = \delta^{-1}(x)$. We claim that there are no finite sets $F \subset \Lambda$ such that:
$$\Lambda = F + \Lambda_x.$$
Indeed, it can be assumed that $\Lambda_x \neq \emptyset$. By the previous lemma, δ is affine. Since $\delta(0) = -1$ and $\delta^{-1}(x) \neq \emptyset$, the function δ takes infinitely many values. However δ takes only finitely many values on $F + \Lambda_x$ for any finite set F. So the claim is proved.

Step 2: First prove that $\mathcal{M}(\beta)$ does not contain a trivial W-submodule. Assume otherwise. There exists $\mu \in \beta$ such that L_μ is a non-zero W-invariant vector.

For any coset γ such that $[L_\mu, \mathcal{M}(\gamma)] \neq 0$, the operator $\mathrm{ad}\, L_\mu$ provides a non-zero W-morphism from $\mathcal{M}(\gamma) \to \mathcal{M}(\beta + \gamma)$. So the previous lemma implies
$$\deg \mathcal{M}(\gamma) = \deg \mathcal{M}(\beta + \gamma).$$
By Lemma 17.2, we have $\delta(\beta + \gamma) = \delta(\beta) + \delta(\gamma) + 1$. Since $\mathcal{M}(\beta)$ is reducible, we get $\delta(\beta) = 0$ or 1. So we get
$$\delta(\beta + \gamma) = \delta(\gamma) + 1, \text{ or } \delta(\beta + \gamma) = \delta(\gamma) + 2.$$
The unique solution of these equations is
$$\delta(\beta) = 0, \ \delta(\gamma) = 0, \text{ and } \delta(\beta + \gamma) = 1.$$
It follows that $\Omega(\mu) \subset \Lambda_1$, where $\Omega(\mu) = \mathrm{Supp}\,[\mathcal{L}, L_\mu]$. By the first step, there are no finite set F such that $\Lambda = F + \Omega(\mu)$, which contradicts Lemma 11.1.

Step 3: Now prove that $\mathcal{M}(\beta)$ does not contain a codimension one W-submodule. Assume otherwise. Let $H \subset \mathcal{M}(\beta)$ be a codimension one W subspace and let μ be the unique element of $\beta \setminus \mathrm{Supp}\, H$. It follows that L_μ^* is a W-invariant vector of the graded dual $\mathcal{L}'$ of $\mathcal{L}$.

For any coset γ such that $[\mathcal{M}(\gamma), \mathcal{M}(\beta - \gamma)] \not\subset H$, the Lie bracket provides a non-zero bilinear map: $\mathcal{M}(\gamma) \times \mathcal{M}(\beta - \gamma) \to \mathcal{M}(\beta)/H \simeq \mathbb{C}$. The previous lemma implies
$$\deg \mathcal{M}(\gamma) = 1 - \deg \mathcal{M}(\beta - \gamma).$$
By Lemma 17.2, we have $\delta(\beta) = \delta(\gamma) + \delta(\beta - \gamma) + 1$. Since $\mathcal{M}(\beta)$ is reducible, we get $\delta(\beta) = 0$ or 1. So we get
$$\delta(\gamma) + \delta(\beta - \gamma) = 0, \text{ or } \delta(\gamma) + \delta(\beta - \gamma) = -1.$$
The unique solution of these equations is
$$\delta(\beta) = 1, \ \delta(\gamma) = 0, \text{ and } \delta(\beta - \gamma) = 0.$$
It follows that $\mathcal{M}(\gamma).L_\mu^* \neq 0$ only if $\delta(\gamma) = 0$. It follows that $\Omega^*(\mu) \subset \Lambda_0 - \mu$, where $\Omega^*(\mu) = \mathrm{Supp}\,\mathcal{L}.L_\mu^*$. By the first step, there are no finite set F such that $\Lambda = F + \Omega^*(\mu)$, which contradicts Lemma 14.9.

Step 4: It follows that the W-module $\mathcal{M}(\beta)$ contains neither a trivial submodule nor a codimension one submodule. By Kaplansky-Santharoubane Theorem, the W-module $\mathcal{M}(\beta)$ is irreducible. $\qquad\square$

18   The quasi-two-cocycle c

In this section, a certain W-equivariant map $\phi : \mathcal{L} \to \mathcal{P}$ is defined. Indeed, ϕ is not a Lie algebra morphism, but we have
$$\phi([L_\lambda, L_\mu]) = c(\lambda, \mu)\{\phi(L_\lambda), \phi(L_\mu)\}$$
for some $c(\lambda, \mu) \in \mathbb{C}^*$. The main result of the section, namely Lemma 18.4, shows that c satisfies a two-cocycle identity. However its validity domain is only a (big) subset of Λ^3. Since c is not an ordinary two-cocycle, it is informally called a "quasi-two-cocycle".

Let $\mathcal{P}$ be the Poisson algebra of symbols of twisted pseudo-differential operators. Recall that

$$\mathcal{P} = \oplus_{(\delta, u) \in Par} \ \Omega_u^\delta$$

Set $A = \mathbb{C}[z, z^{-1}]$, $A_u = z^u A$ for $u \in \mathbb{C}/\mathbb{Z}$. As before, it will be convenient to represent elements in Ω_u^δ as $f\partial^{-\delta}$, where $f \in A_{u-\delta}$. Relative to the Poisson bracket, we have $\{\Omega_u^\delta, \Omega_v^\eta\} \subset \Omega_{u+v}^{\delta+\eta+1}$.

Let (δ, u), (η, v) and $(\tau, w) \in Par$.

Lemma 18.1. *Let a, b, c be three scalars. Assume that*
$$a\{X, \{Y, Z\}\} + b\{Z, \{X, Y\}\} + c\{Y, \{Z, X\}\} = 0$$
for any $(X, Y, Z) \in \Omega_u^{-\delta} \times \Omega_v^{-\eta} \times \Omega_w^{-\tau}$

If none of the four couples (η, τ), $(\delta, \eta + \tau - 1)$, (η, δ), $(\tau, \delta + \eta - 1)$ is $(0, 0)$, then we have
$$a = b.$$

Proof. Assume that none of the four couples is $(0, 0)$. Define three maps $\mu_i : \Omega_u^{-\delta} \times \Omega_v^{-\eta} \times \Omega_w^{-\tau} \to \Omega_{u+v+w}^{-\delta-\eta-\tau+2}$, $(X, Y, Z) \mapsto \mu_i(X, Y, Z)$ as follows:
$$\mu_1(X, Y, Z) = \{X, \{Y, Z\}\}$$
$$\mu_2(X, Y, Z) = \{Z, \{X, Y\}\}$$
$$\mu_3(X, Y, Z) = \{Y, \{Z, X\}\}$$
Set $X = f\partial^\delta$, $Y = g\partial^\eta$ and $Z = h\partial^\tau$, where $f \in A_{u+\delta}$, $g \in A_{v+\eta}$ and $h \in A_{w+\tau}$. We have:

$$\begin{aligned}
&\mu_1(X, Y, Z) \\
&= \{f\partial^\delta, \{g\partial^\eta, h\partial^\tau\}\} \\
&= \{f\partial^\delta, (\eta gh' - \tau g'h)\partial^{\eta+\tau-1}\} \\
&= [\delta\eta fgh'' - \delta\tau fg''h + (\delta\eta - \delta\tau)fg'h' - (\eta + \tau - 1)\eta f'gh' \\
&\quad + (\eta + \tau - 1)\tau f'g'h]\partial^{\delta+\eta+\tau-2}.
\end{aligned}$$

So $[a\mu_1 + b\mu_2 + c\mu_3](X, Y, Z)$ can be expressed as:

$$(Afgh'' + Bfg''h + Cf''gh + Dfg'h' + Ef'gh' + Ef'g'h)\partial^{\delta+\eta+\tau-2},$$
where the six coefficients are given by:

$A = (a-c)\delta\eta$

$B = (b-a)\tau\delta$

$C = (c-b)\eta\tau$

$D = [a(\eta-\tau) - b(\delta+\eta-1) + c(\tau+\delta-1)]\delta$

$E = [-a(\tau+\eta-1) + b(\delta+\eta-1) + c(\tau-\delta)]\eta$

$F = [a(\tau+\eta-1) + b(\tau-\eta) - c(\delta+\tau-1)]\tau.$

The equation $a\mu_1 + b\mu_2 + c\mu_3 = 0$ implies that the six coefficients $A, B, \ldots, F$ are all zero.

If $\delta\tau \neq 0$, the equality $a = b$ follows from $B = 0$. Assume otherwise. Since δ or τ is zero, we have $\eta \neq 0$.

If $\tau \neq 0$ but $\delta = 0$, it follows from $C = 0$ that $b = c$. Therefore the identity $E = 0$ implies that $(b-a)(\tau+\eta-1) = 0$. Since $\delta = 0$, we get $\tau+\eta-1 \neq 0$ and thus $a = b$.

The case $\tau = 0$ but $\delta \neq 0$ is strictly similar.

In the case $\tau = \delta = 0$, from $E = 0$ we get $(b-a)(\eta-1) = 0$. Since $(\delta, \eta+\tau-1) = (0, \eta-1)$, it follows that $\eta-1 \neq 0$ and therefore $a = b$. $\qquad\square$

Let $\delta \in \mathbb{C}$, let $u, v, w \in \mathbb{C}/\mathbb{Z}$ and let $\theta : \Omega_u^0 \times \Omega_v^0 \to \Omega_{u+v}^1$ be any W-equivariant bilinear map.

Lemma 18.2. *Let a, b, c be three scalars, with $a \neq 0$. Assume that*
$$a\{X, \theta(Y, Z)\} + b\{Z, \{X, Y\}\} + c\{Y, \{Z, X\}\} = 0$$
for any $(X, Y, Z) \in \Omega_u^{-\delta} \times \Omega_v^0 \times \Omega_w^0$. Then we have $\theta = 0$.

Proof. Define three maps $\mu_i : \Omega_u^{-\delta} \times \Omega_v^0 \times \Omega_w^0 \to \Omega_{u+v+w}^{-\delta+2}, (X, Y, Z) \mapsto \mu_i(X, Y, Z)$ as follows:

$\mu_1(X, Y, Z) = \{X, \theta(Y, Z)\}$

$\mu_2(X, Y, Z) = \{Z, \{X, Y\}\}$

$\mu_3(X, Y, Z) = \{Y, \{Z, X\}\}$

As before, identify Ω_s^0 with A_s. By Lemma 13.2, there are two constants A, B such that $\theta(g, h) = Ah\mathrm{d}\,g + Bg\mathrm{d}\,h$. In term of symbols, we get $\theta(g, h) = [Ag'h + Bgh']\partial^{-1}$.

Set $X = f\partial^\delta$, $Y = g$ and $Z = h$, where $f \in A_{u+\delta}$, $g \in A_v$ and $h \in A_w$. We have:

$$\begin{aligned}
\mu_1(X, Y, Z) &= \{f\partial^\delta, [Ag'h + Bgh']\partial^{-1}\} \\
&= [A\delta fg''h + B\delta fgh'' + \delta(A+B)fg'h' + Af'g'h + Bf'gh']\partial^{\delta-2}.\\
\mu_2(X, Y, Z) &= \{h, \{f\partial^\delta, g\}\} \\
&= -\delta(\delta-1)fh'g'\partial^{\delta-2} \\
\mu_3(X, Y, Z) &= \{g, \{h, f\partial^\delta\}\} \\
&= \delta(\delta-1)fh'g'\partial^{\delta-2}.
\end{aligned}$$

The coefficient of the monomial $f'g'h$ and the monomial $f'gh'$ in the expression $(a\mu_1 + b\mu_2 + c\mu_3)(X, Y, Z)$ are respectively A and B. Thus $A = B = 0$ and so θ vanishes. $\square$

Let $\mathcal{L}$ be a non-integrable Lie algebra in the class $\mathcal{G}$. As before, fix once for all a primitive element $\alpha \in \Lambda$ such that $l(\alpha) \neq 0$. One can normalize L_0 in a way that $l(\alpha) = 1$.

By Lemmas 11.4 and 14.8, the Lie subalgebra $\mathcal{L}(\alpha) = \oplus_{n\in\mathbb{Z}} \mathcal{L}_{n\alpha}$ is isomorphic to W. For any $\mathbb{Z}\alpha$-coset β, set $\mathcal{M}(\beta) = \oplus_{\gamma\in\beta} \mathcal{L}_\gamma$. For such β, denote by $s(\beta)$ the spectrum of L_0 on $\mathcal{M}(\beta)$. It is clear that $s(\beta)$ is exactly a $\mathbb{Z}$-coset, therefore $s(\beta)$ is a well defined element of $\mathbb{C}/\mathbb{Z}$.

By Lemma 17.4, the W-module $\mathcal{M}(\beta)$ is irreducible, and by Kaplansky-Santharoubane Theorem, there is an isomorphism of W-modules
$$\phi_\beta : \mathcal{M}(\beta) \to \Omega^{\delta(\beta)}_{s(\beta)}.$$
Fix once for all the isomorphism ϕ_β for all $\beta \in \Lambda/\mathbb{Z}\alpha$. Let $\phi : \mathcal{L} \to \mathcal{P}$ be the map whose restriction to each $\mathcal{M}(\beta)$ is ϕ_β. Thus ϕ is a morphism of W-modules. As it will be seen later on, some modification of ϕ will be a morphism of Lie algebras.

Let β, γ be two $\mathbb{Z}\alpha$-cosets. It is clear that $[\mathcal{M}(\beta), \mathcal{M}(\gamma)] \subset \mathcal{M}(\beta + \gamma)$, so the Lie bracket provides a morphism of W-modules
$$[\,,\,] : \mathcal{M}(\beta) \times \mathcal{M}(\gamma) \to \mathcal{M}(\beta + \gamma).$$
By Lemma 17.2, we have $\delta(\beta + \gamma) = \delta(\beta) + \delta(\gamma) + 1$. Therefore the bracket of symbols provides another morphism of W-modules:
$$\{\,,\,\} : \Omega^{\delta(\beta)}_{s(\beta)} \times \Omega^{\delta(\gamma)}_{s(\gamma)} \to \Omega^{\delta(\beta+\gamma)}_{s(\beta+\gamma)}.$$
Thus we get a diagram of W-modules :

$$
\begin{array}{ccc}
\mathcal{M}(\beta) \times \mathcal{M}(\gamma) & \xrightarrow{\ [\,,\,]\ } & \mathcal{M}(\beta + \gamma) \\
\Big\downarrow{\scriptstyle \phi_\beta \times \phi_\gamma} & & \Big\downarrow{\scriptstyle \phi_{\beta+\gamma}} \\
\Omega^{\delta(\beta)}_{s(\beta)} \times \Omega^{\delta(\gamma)}_{s(\gamma)} & \xrightarrow{\ \{\,,\,\}\ } & \Omega^{\delta(\beta+\gamma)}_{s(\beta+\gamma)}
\end{array}
$$

This diagram is almost commutative:

Lemma 18.3. *There exists a function $c : \Lambda/\mathbb{Z}\alpha \times \Lambda/\mathbb{Z}\alpha \to \mathbb{C}^*$ such that*
$$\phi([X,Y]) = c(\beta,\gamma)\{\phi(X), \phi(Y)\}$$
for any β, $\gamma \in \Lambda/\mathbb{Z}\alpha$ and any $X, Y \in \mathcal{M}(\beta) \times \mathcal{M}(\gamma)$. Moreover
$$c(\beta,\gamma) = c(\gamma,\beta)$$
if $(\delta(\beta), \delta(\gamma)) \neq (0,0)$.

Proof. Let $\beta, \gamma \in \Lambda/\mathbb{Z}\alpha$.

First consider the case $(\delta(\beta), \delta(\gamma)) \neq (0,0)$. By Lemma 13.2, any W-equivariant bilinear map $\Omega^{\delta(\beta)}_{s(\beta)} \times \Omega^{\delta(\gamma)}_{s(\gamma)} \to \Omega^{\delta(\beta+\gamma)}_{s(\beta+\gamma)}$ is proportional to the Poisson bracket of symbols. Thus there is a constant $c(\beta,\gamma) \in \mathbb{C}$ such that $\phi([X,Y]) = c(\beta,\gamma)\{\phi(X), \phi(Y)\}$ for any $X, Y \in \mathcal{M}(\beta) \times \mathcal{M}(\gamma)$. Moreover, by Lemma 17.1,

59

we have $[\mathcal{M}(\beta), \mathcal{M}(\gamma)] \neq 0$. Therefore $c(\beta, \gamma) \neq 0$. Since the Poisson bracket and the Lie bracket are skew symmetric, we also have $c(\beta, \gamma) = c(\gamma, \beta)$.

Next consider the case $(\delta(\beta), \delta(\gamma)) = (0, 0)$. We claim that
$$[\mathcal{M}(\beta), \mathcal{M}(\gamma)] = 0.$$

In order to prove the claim, we may assume that δ takes the value 0. Define the W-equivariant bilinear map $\theta : \Omega_{s(\beta)}^{\delta(\beta)} \times \Omega_{s(\gamma)}^{\delta(\gamma)} \to \Omega_{s(\beta+\gamma)}^{\delta(\beta+\gamma)}$ by the requirement: $\phi_{\beta+\gamma}([X, Y]) = \theta(\phi_\beta(X), \phi_\gamma(Y))$.

Since $\delta(0) = -1$, δ takes the value 0 and δ is affine, there is some $\xi \in \Lambda/\mathbb{Z}\alpha$ such that $\delta(\xi) \notin \{0, -1\}$. Let $X, Y, Z \in \mathcal{M}(\beta) \times \mathcal{M}(\gamma) \times \mathcal{M}(\xi)$. Since $\delta(\xi) + 1 \neq 0$, we get
$$\phi([X, [Y, Z]]) = c(\beta, \gamma + \xi)\{\phi(X), \phi([Y, Z])\}.$$
$$= c(\beta, \gamma + \xi)c(\gamma, \xi)\{\phi(X), \{\phi(Y), \phi(Z)\}\}$$

Similarly, we have
$$\phi([Y, [Z, X]]) = c(\gamma, \beta + \xi)c(\beta, \xi)\{\phi(Y), \{\phi(Z), \phi(X)\}\}, \text{ and}$$
$$\phi([Z, [X, Y]]) = c(\xi, \alpha + \beta)\{\phi(Z), \theta(X, Y)\}.$$
Thus the Jacobi identity in $\mathcal{L}$: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ implies that
$$c(\beta, \gamma + \xi)c(\gamma, \xi)\{X, \{Y, Z\}\} + c(\gamma, \beta + \xi)c(\beta, \xi)\{Y, \{Z, X\}\}$$
$$+ c(\xi, \alpha + \beta)\{Z, \theta(X, Y)\} = 0,$$
for any $(X, Y, Z) \in \Omega_{s(\beta)}^0 \times \Omega_{s(\gamma)}^0 \times \Omega_{s(\xi)}^{\delta(\xi)}$. Since $c(\xi, \alpha + \beta)$ is not zero, it follows from Lemma 18.2 that θ vanishes. Thus the claim is proved.

Since $[\mathcal{M}(\beta), \mathcal{M}(\gamma)] = 0$ and $\{\Omega_{s(\beta)}^0, \Omega_{s(\gamma)}^0\} = 0$, any value $c(\beta, \gamma) \in \mathbb{C}^*$ is suitable. $\qquad\square$

Lemma 18.4. *Let β, γ and η be three $\mathbb{Z}\alpha$-cosets. If none of the four couples $(\delta(\beta), \delta(\gamma))$, $(\delta(\beta + \gamma), \delta(\eta))$, $(\delta(\gamma), \delta(\eta))$ and $(\delta(\beta), \delta(\gamma + \eta))$ is $(0, 0)$, then we have:*
$$c(\beta, \gamma)c(\beta + \gamma, \eta) = c(\beta, \gamma + \eta)c(\gamma, \eta).$$

Proof. As in the previous proof, we have:
$$\phi([X, [Y, Z]]) = c(\beta, \gamma + \eta)c(\gamma, \eta)\{\phi(X), \{\phi(Y), \phi(Z)\}\}.$$
for any $(X, Y, Z) \in \mathcal{M}(\beta) \times \mathcal{M}(\gamma) \times \mathcal{M}(\eta)$. It follows from the Jacobi identity in $\mathcal{L}$ that
$$c(\beta, \gamma + \eta)c(\gamma, \eta)\{X, \{Y, Z\}\} + c(\eta, \beta + \gamma)c(\beta, \gamma)\{Z, \{X, Y\}\}$$
$$+ c(\gamma, \eta + \beta)c(\gamma, \eta)\{Y, \{Z, X\}\} = 0,$$
for any $(X, Y, Z) \in \Omega_{s(\beta)}^{\delta(\beta)} \times \Omega_{s(\gamma)}^{\delta(\gamma)} \times \Omega_{s(\eta)}^{\delta(\eta)}$. Thus the lemma follows from Lemma 18.1. $\qquad\square$

Remark. *The equation satisfied by c is exactly the equation of a two-cocycle of $\Lambda/\mathbb{Z}\alpha$ with values in $\mathbb{C}^*$, except that its validity domain is a subset of $(\Lambda/\mathbb{Z}\alpha)^3$. So c is a "quasi-two-cocycle".*

19   Proof of Theorem 3

As before, let $\mathcal{L} \in \mathcal{G}$ be a non-integrable Lie algebra. Fix once for all a primitive element $\alpha \in \Lambda$ such that $l(\alpha) \neq 0$ and recall that $\mathcal{L}(\alpha) \simeq W$. As before,

we normalize L_0 in such a way that $l(\alpha) = 1$. In the previous section, a W-equivariant map $\phi : \mathcal{L} \to \mathcal{P}$ has been defined. The map ϕ is not a Lie algebra morphism, but the defect is accounted by a map $c : \Lambda/\mathbb{Z}\alpha \times \Lambda/\mathbb{Z}\alpha \to \mathbb{C}^*$.

In this section, it is proved that c is indeed a "quasi-boundary". This allows us to modify ϕ to get an algebra morphism $\psi : \mathcal{L} \to \mathcal{P}$, from which Theorem 3 is deduced.

Lemma 19.1. *Let M be a lattice and let $R = \oplus_{m \in M} R_m$ be a commutative associative M-graded algebra satisfying the following conditions:*

(i) $\dim R_m \leq 1$ *for all $m \in M$,*

(ii) $X.Y \neq 0$ *for any two non-zero homogeneous elements X, Y of R.*

Then there exists an algebra morphism $\chi : R \to \mathbb{C}$ with $\chi(X) \neq 0$ for any non-zero homogeneous element $X \in R$.

Proof. Let S be the set of non-zero homogeneous elements of R. By hypothesis, S is a multiplicative subset. Since $S/\mathbb{C}^* \simeq \operatorname{Supp} R$ is countable, the algebra R_S has countable dimension, therefore any maximal ideal of R_S provides an algebra morphism $\chi : R_S \to \mathbb{C}$. So its restriction to R is an algebra morphism $\chi : R \to \mathbb{C}$ with $\chi(X) \neq 0$ for any $X \in S$. $\qquad\square$

Let M be a lattice, and let $d : M \to \mathbb{C}$ be an additive map. Let $c : M \times M \to \mathbb{C}^*$, $(l, m) \mapsto c(l, m)$ be a function.

Lemma 19.2. *Assume the following hypotheses for all $l, m, n \in M$:*

(i) $c(l, m) = c(m, l)$ *whenever $(d(l), d(m)) \neq (-1, -1)$.*

(ii) $c(l, m)c(l + m, n) = c(l, m + n)c(m, n)$ *whenever none of the four couples $(d(l), d(m))$, $(d(l+m), d(n))$, $(d(l), d(m+n))$ and $(d(m), d(n))$ is $(-1, -1)$.*

Then there exists a function $b : M \to \mathbb{C}^*$ such that
$$c(l, m) = b(l)b(m)/b(l + m)$$
for any couple (l, m) with $(d(l), d(m)) \neq (-1, -1)$.

Proof. Define an algebra structure on $\mathbb{C}[M]$ by the formula:
$$e^l * e^m = c(l, m)e^{l+m}.$$
Set $N = \{m \in M \,|\, \Re d(m) \geq 0\}$, where the notation $\Re z$ means the real part of the complex number z and let $\mathbb{C}[N]$ be the subalgebra of $\mathbb{C}[M]$ with basis $(e^n)_{n \in N}$. Since $d(n) \neq -1$ for any $n \in N$, it follows from Identities (i) and (ii) that $\mathbb{C}[N]$ is a commutative and associative algebra. By the previous lemma, there is an algebra morphism $\chi : \mathbb{C}[N] \to \mathbb{C}$ with $\chi(X) \neq 0$ for any element $X \in S$, where S denotes the set of non-zero homogeneous elements in $\mathbb{C}[N]$.

By restriction, the product $*$ defines a bilinear map $\beta : \mathbb{C}[N] \times \mathbb{C}[M] \to \mathbb{C}[M]$. There are no couples $(n, m) \in N \times M$ with $(d(n), d(m)) = (-1, -1)$. Therefore β is a structure of $\mathbb{C}[N]$-module on $\mathbb{C}[M]$. Indeed for $n_1, n_2 \in N$ and $m \in M$, we have $e^{n_1} * (e^{n_2} * e^m) = (e^{n_1} * e^{n_2}) * e^m$.

As each $X \in S$ acts bijectively on $\mathbb{C}[M]$, $\mathbb{C}[M]$ contains $\mathbb{C}[N]_S$. Since $M = N - N$, $\mathbb{C}[M]$ is isomorphic to $\mathbb{C}[N]_S$ as a $\mathbb{C}[N]$-module. Therefore χ extends to a morphism of $\mathbb{C}[N]$-modules $\chi : \mathbb{C}[M] \to \mathbb{C}$. So it satisfies
$$\chi(e^n * e^m) = \chi(e^n)\chi(e^m)$$
for any couple $(n, m) \in N \times M$.

Next, prove the stronger assertion that
$$\chi(e^m * e^l) = \chi(e^m)\chi(e^l)$$
for any couple (m, l) with $(d(m), d(l)) \neq (-1, -1)$. Choose any $n \in N$ with $\mathbb{R}d(m + n) \geq 0$. Since $n \in N$, it follows from the previous identity that:
$$\chi(e^n * (e^m * e^l)) = \chi(e^n)\chi(e^m * e^l).$$
Similarly, it follows from the fact that $n \in N$ and $n + m \in N$ that
$$\chi((e^n * e^m) * e^l) = \chi(e^n * e^m)\chi(e^l) = \chi(e^n)\chi(e^m)\chi(e^l)$$
By hypothesis $(d(m), d(l)) \neq (-1, -1)$, and each couple $(d(n), d(m+l))$, $(d(n), d(m))$ and $(d(n + m), d(l))$ contains a scalar with non-negative real part. So none of these couples is $(-1, -1)$, and therefore we have
$$e^n * (e^m * e^l) = (e^n * e^m) * e^l$$
It follows that $\chi(e^m * e^l) = \chi(e^m)\chi(e^l)$ and therefore we have
$$c(m, l)\chi(e^{m+l}) = \chi(e^m)\chi(e^l),$$
for any (m, l) with $(d(m), d(l)) \neq (-1, -1)$. Thus the function $b(m) = \chi(e^m)$ satisfies the required identity. $\qquad\square$

Remark. *It follows that a symmetric two-cocycle of M with value in $\mathbb{C}^*$ is a boundary. This corollary is obvious. Indeed a symmetric two-cocycle gives rise to a central extension*
$$1 \to \mathbb{C}^* \to \widehat{M} \to M \to 0$$
where $\widehat{M}$ is an abelian group. It is split because M is free in the category of abelian groups.

Let $b : \Lambda/\mathbb{Z}\alpha \to \mathbb{C}^*$ be a function. Recall that $\mathcal{L} = \oplus_{\beta \in \Lambda/\mathbb{Z}\alpha} \mathcal{M}(\beta)$. Define a new morphism of W-modules $\psi : \mathcal{L} \to \mathcal{P}$ by the formula:
$$\psi(X) = b(\beta)\phi(X)$$
for any $X \in \mathcal{M}(\beta)$ and any $\beta \in \Lambda/\mathbb{Z}\alpha$.

Lemma 19.3. *There exists a function $b : \Lambda/\mathbb{Z}\alpha \to \mathbb{C}^*$ such that $\psi : \mathcal{L} \to \mathcal{P}$ is a morphism of Lie algebras.*

Proof. Set $M = \Lambda/\mathbb{Z}\alpha$ and for $\beta \in M$, set $d(\beta) = -1 - \delta(\beta)$. It follows from Lemma 17.2 that the map $d : M \to \mathbb{C}$ is additive. By Lemmas 18.3 and 18.4, the quasi-two-cocycle c and the additive map d satisfies the hypothesis of the previous lemma. Therefore there exists a function $b : \Lambda/\mathbb{Z}\alpha \to \mathbb{C}^*$ such that
$$c(\beta, \gamma) = b(\beta)b(\gamma)/b(\beta + \gamma)$$
for any couple (β, γ) with $(\delta(\beta), \delta(\gamma)) \neq (0, 0)$.

Choose such a function b. We claim that $\psi([X, Y]) = \{\psi(X), \psi(Y)\}$ for any $X \in \mathcal{M}(\beta)$, $Y \in \mathcal{M}(\gamma)$ and any β, γ be in $\Lambda/\mathbb{Z}\alpha$.

First assume that $(\delta(\beta), \delta(\gamma)) \neq (0,0)$. We have

$$\begin{aligned}
\psi([X,Y]) &= b(\beta + \gamma)\,\phi([X,Y]) \\
&= b(\beta + \gamma)c(\beta, \gamma)\,\{\phi(X), \phi(Y)\} \\
&= b(\beta)b(\gamma)\,\{\phi(X), \phi(Y)\}.
\end{aligned}$$

Thus $\psi([X,Y]) = \{\psi(X), \psi(Y)\}$.

Consider now the case $(\delta(\beta), \delta(\gamma)) = (0,0)$. Since $\{\Omega_u^0, \Omega_s^0\} = 0, \forall u,\ s \in \mathbb{C}/\mathbb{Z}$ it follows that $\psi([X,Y]) = \{\psi(X), \psi(Y)\}$ because both sides of the identity are zero.

Therefore ψ is an algebra morphism. $\qquad\square$

Define the map $\pi : \Lambda \to \mathbb{C}^2$ by the formula:
$$\pi(\lambda) = (l(\lambda) - \delta(\lambda) - 1, -1 - \delta(\lambda)).$$
it follows from Lemma 17.2 that π is additive. Since $\mathcal{P}$ is $\mathbb{C}^2$-graded, one can define the Λ-graded Lie algebra $\pi^*\mathcal{P}$. When π is one-to-one, $\pi^*\mathcal{P}$ is the Lie algebra W_π defined in the introduction. In general, the notation π^* has been defined in Section 1.6 and W_π in Section 12.7.

Lemma 19.4. *We have $\mathcal{L} \simeq \pi^*\mathcal{P}$.*

Proof. Let $\lambda \in \Lambda$. By defintion, $\psi(L_\lambda) = f\partial^{-\delta(\lambda)}$, for some twisted function f. Moreover $[L_0, L_\lambda] = l(\lambda)L_\lambda$, therefore we have $z\frac{d}{dz}f = [l(\lambda) - \delta(\lambda)]f$, therefore f is proportional to $z^{l(\lambda) - \delta(\lambda)}$. It follows that ψ maps isomorphically $\mathcal{L}_\lambda$ to $\mathcal{P}_{\pi(\lambda)}$. By Lemma 1.1, $\mathcal{L}$ is precisely $\pi^*\mathcal{P}$. $\qquad\square$

Recall that the condition $\mathcal{C}$ is:
$$\operatorname{Im}\pi \not\subset \mathbb{C}\rho \text{ and } 2\rho \notin \operatorname{Im}\pi.$$

Theorem 3. *Let Λ be a lattice.*

(i) *If $\mathcal{L} \in \mathcal{G}$ is a primitive non-integrable Lie algebra, then there is an injective additive map $\pi : \Lambda \to \mathbb{C}^2$ satisfying condition $\mathcal{C}$ such that $\mathcal{L} \simeq W_\pi$.*

(ii) *Conversely, if $\pi : \Lambda \to \mathbb{C}^2$ is injective and satisfies condition $\mathcal{C}$, then the Lie algebra W_π is simple (and, in particular, it is primitive).*

Proof. By the previous lemma, $\mathcal{L} \simeq \pi^*\mathcal{P} = W_\pi$. Set $M = \operatorname{Ker}\pi$. There is a sublattice Λ_1 such that $\Lambda = M \oplus \Lambda_1$. It follows that $W_\pi \simeq \mathbb{C}[M] \otimes W_{\pi_1}$, where $\pi_1 : \Lambda_1 \to \mathbb{C}$ is the restriction of π to Λ_1. Since $\mathcal{L}$ is primitive, it follows that $M = 0$, hence π is injective.

If $\pi(\Lambda) \subset \mathbb{C}\rho$, then W_π is abelian. If $2\rho \in \pi(\Lambda)$, then $E_{-2\rho}$ belongs to W_π and therefore $W_\pi \neq [W_\pi, W_\pi]$. Since $\mathcal{L}$ is simple graded, π satisfies the condition $\mathcal{C}$.

The converse follows from Lemma 12.3. $\qquad\square$

Remark. *Set $\omega = (1,0)$, so that E_ω is the symbol of $z^2\partial$. It follows from the proof that, for any primitive vector $\alpha \in \Lambda$ with $l(\alpha) \neq 0$, there exists a unique $\pi_\alpha : \Lambda \to \mathbb{C}^2$ such that $\pi_\alpha(\alpha) = \omega$ and $\mathcal{L} \simeq W_{\pi_\alpha}$. Therefore, there are many injective additive maps π such that $\mathcal{L} \simeq W_\pi$ and some of them do not contain ω in their image.*

References

[BZ]   Georgia Benkart and Efim Zelmanov: Lie algebras graded by root systems, in *Non-associative algebras and its applications*, Kluwer Acad. Publ., Math. Appl., 303 (1994) 31-38.

[C]   Vyjayanthi Chari: Integrable representations of affine Lie-algebras. Invent. Math. 85 (1986) 317-335.

[CP]   Vyjayanthi Chari and Andrew Pressley: New unitary representations of loop groups. Math. Ann. 275 (1986) 87-104.

[D]   Jacques Dixmier: *Algèbres enveloppantes*, Cahiers scientifiques XXXVII, Gauthier-Villars, Paris (1974).

[E]   Alberto Elduque: Fine gradings on simple classical Lie algebras. J. Algebra 324 (2010) 3532-3571.

[Gr]   Ya Grozman: Classification of Bilinear Invariant Operators on Tensor Fields. Funct. anal. Appl. 14 (1980) 58-59.

[I]   Kenji Iohara and Yoshiyuki Koga: *Representation theory of the Virasoro algebra,* Springer Monographs in Mathematics, Springer-Verlag (2011).

[IM]   Kenji Iohara and Olivier Mathieu: A Global Version of Grozman's Theorem, in preparation.

[J]   N. Jacobson: *Structure and Representations of Jordan Algebras*, Am. Math. Soc., Colloquium Publications 39 (1968).

[Ka1]   V.G. Kac: Simple irreducible graded Lie algebras of finite growth. Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968) 1323-1367.

[Ka2]   V.G. Kac: Some problems on infinite dimensional Lie algebras, in *Lie algebras and related topics*, Springer-Verlag, Lect. Notes in Math. 933 (1982) 117-126.

[Kan]   I.M. Kantor: Models of the exceptional Lie algebras. Dokl. Akad. Nauk SSSR 208 (1973), 1276-1279.

[Kap]   Irving Kaplansky, The Virasoro algebra. Comm. Math. Phys. 86 (1982), 49-54.

[KS]   Irving Kaplansky and L. J. Santharoubane: Harish-Chandra modules over the Virasoro algebra. in *Infinite-dimensional groups with applications*, Math. Sci. Res. Inst. Publ. 4 (1985) 217-231.

[Ki]   A.A. Kirillov: *The Orbit Method.* Notes from the European scholl of Group Theory, Luminy (1991)

[Ko]   Max Koecher: Imbedding of Jordan algebras into Lie algebras. American Journal of Mathematics 89 (1967) 787-816

[MP]   Christiane Martin and Alain Piard: Classification of the indecomposable bounded admissible modules over the Virasoro Lie algebra with weightspaces of dimension not exceeding two. Comm. Math. Phys. 150 (1992) 465-493.

[M1]   O. Mathieu: Sur un problème de V.G. Kac: la classification de certaines algèbres de Lie graduées simples. J. of Algebra 102 (505-536) 1986.

[M2]   O. Mathieu: Classification des algèbres de Lie de croissance ≤ 1. Inv. Math 108 (1992) 455-519.

[MS]   A. Meurman and L.J. Santharoubane: Cohomology and Harish-Chandra modules over the Virasoro algebra. Comm. Algebra 16 (1988) 27-35.

[Mi]   W. Miller: *On Lie algebras and some special functions of mathematical physics.* Mem. Amer. Math. Soc. 50 (1964)

[Mo]   Robert Moody: A new class of Lie algebras. J. Algebra 10 (1968) 211-230.

[T]   Jacques Tits: Une classe d'algèbres de Lie en relation avec les algèbres de Jordan. Indag. Math. 24 (1962) 530-535.

[Y]   R. Yu : Algèbre de Lie de type Witt. Comm. in Alg. 25 (1997) 1471-1484.

Authors addresses:
   Université Claude Bernard Lyon 1,
   Institut Camille Jordan, UMR 5028 du CNRS
   43, bd du 11 novembre 1918
   69622 Villeurbanne Cedex
   FRANCE
   Email addresses:
   iohara@math.univ-lyon1.fr
   mathieu@math.univ-lyon1.fr